

THE ROHLIN PROPERTY FOR COACTIONS OF FINITE DIMENSIONAL C^* -HOPF ALGEBRAS ON UNITAL C^* -ALGEBRAS

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ABSTRACT. We shall introduce the approximate representability and the Rohlin property for coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra and discuss some basic properties of approximately representable coactions and coactions with the Rohlin property of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra. Also, we shall give an example of an approximately representable coaction of a finite dimensional C^* -Hopf algebra on a simple unital C^* -algebra which has also the Rohlin property and we shall give the 1-cohomology vanishing theorem for coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra and the 2-cohomology vanishing theorem for twisted coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra. Furthermore, we shall introduce the notion of the approximately unitary equivalence of coactions of a finite dimensional C^* -Hopf algebra H on a unital C^* -algebra A and show that if ρ and σ , coactions of H on a separable unital C^* -algebra A , which have the Rohlin property, are approximately unitarily equivalent, then there is an approximately inner automorphism α on A such that

$$\sigma = (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}.$$

1. INTRODUCTION

Let A be a unital C^* -algebra and H a finite dimensional C^* -Hopf algebra with the comultiplication Δ . In this paper, we shall introduce the approximate representability and the Rohlin property for coactions of H on A and discuss some basic properties of approximately representable coactions and coactions with the Rohlin property of H on A . Also, we shall give an example of an approximately representable coaction of a finite dimensional C^* -Hopf algebra on a simple unital C^* -algebra which has also the Rohlin property and we shall give the following 1-cohomology vanishing theorem: Let ρ be a coaction of H on A with the Rohlin property. Let v be a unitary element in $A \otimes H$ with

$$(v \otimes 1)(\rho \otimes \text{id})(v) = (\text{id} \otimes \Delta)(v)$$

and let σ be the coaction of H on A defined by $\sigma = \text{Ad}(v) \circ \rho$. Then there is a unitary element $x \in A$ such that

$$\sigma = \text{Ad}(x \otimes 1) \circ \rho \circ \text{Ad}(x^*).$$

Furthermore, we shall give the following 2-cohomology vanishing theorem: Let (ρ, u) be a twisted coaction of H^0 on A with the Rohlin property. Then there is a unitary element $x \in A \otimes H$ such that

$$(x \otimes 1)(\rho \otimes \text{id})(x)u(\text{id} \otimes \Delta)(x)^* = 1 \otimes 1 \otimes 1.$$

Finally, we shall introduce the notion of the approximately unitary equivalence of coactions of H and show that if ρ and σ , coactions of H on a separable unital C^* -algebra A , which have the Rohlin property, are approximately unitarily equivalent,

then there is an approximately inner automorphism α on A such that

$$\sigma = (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}.$$

For an algebra X , we denote by 1_X and id_X the unit element in X and the identity map on X , respectively. If no confusion arises, we denote them by 1 and id , respectively.

For projections p, q in a C^* -algebra C , we write $p \sim q$ in C if p is Murray-von Neumann equivalent to q in C .

For each $n \in \mathbf{N}$, we denote by $M_n(\mathbf{C})$ the $n \times n$ -matrix algebra over \mathbf{C} and I_n denotes the unit element in $M_n(\mathbf{C})$.

2. PRELIMINARIES

Let H be a finite dimensional C^* -Hopf algebra. We denote its comultiplication, counit and antipode by Δ , ϵ and S . We shall use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in H$ which suppresses a possible summation when we write the comultiplications. We denote by N the dimension of H . Let H^0 be the dual C^* -Hopf algebra of H . We denote its comultiplication, counit and antipode by Δ^0 , ϵ^0 and S^0 . There is the distinguished projection e in H . We note that e is the Haar trace on H^0 . Also, there is the distinguished projection τ in H^0 which is the Haar trace on H .

Throughout this paper, H denotes a finite dimensional C^* -Hopf algebra and H^0 its dual C^* -Hopf algebra. Since H is finite dimensional, $H \cong \bigoplus_{k=1}^K M_{d_k}(\mathbf{C})$ as C^* -algebras. Let $\{v_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ be a system of matrix units of H . Let $\{w_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ be a basis of H satisfying Szymański and Peligrad [10, Theorem 2.2,2]. We call it a system of *co-matrix units* of H . Also, let $\{\phi_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ and $\{\omega_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ be systems of matrix units and comatrix units of H^0 , respectively. Furthermore, let ρ_H^A be the trivial coaction of H on A defined by $\rho_H^A(a) = a \otimes 1$ for any $a \in A$.

Following Masuda and Tomatsu [7], we shall define a twisted coaction of H on A and its exterior equivalence.

Definition 2.1. Let ρ be a weak coaction of H on A and u a unitary element in $A \otimes H \otimes H$. The pair (ρ, u) is a *twisted coaction* of H on A if the following conditions hold:

- (1) $(\rho \otimes \text{id}) \circ \rho = \text{Ad}(u) \circ (\text{id} \otimes \Delta) \circ \rho$,
- (2) $(u \otimes 1)(\text{id} \otimes \Delta \otimes \text{id})(u) = (\rho \otimes \text{id} \otimes \text{id})(u)(\text{id} \otimes \text{id} \otimes \Delta)(u)$,
- (3) $(\text{id} \otimes \phi \otimes \epsilon)(u) = (\text{id} \otimes \epsilon \otimes \phi)(u) = \phi(1)1$ for any $\phi \in H^0$.

Definition 2.2. For $i = 1, 2$, let (ρ_i, u_i) be a twisted coaction of H on A . We say that (ρ_1, u_1) is *exterior equivalent* to (ρ_2, u_2) if there is a unitary element v in $A \otimes H$ satisfying following conditions:

- (1) $\rho_2 = \text{Ad}(v) \circ \rho_1$,
- (2) $u_2 = (v \otimes 1)(\rho_1 \otimes \text{id})(v)u_1(\text{id} \otimes \Delta)(v^*)$.

By routine computations, $(\text{id} \otimes \epsilon)(v) = 1$ and the above equivalence is an equivalence one. We write $(\rho_1, u_1) \sim (\rho_2, u_2)$ if (ρ_1, u_1) is exterior equivalent to (ρ_2, u_2) .

Remark 2.1. Let (ρ, u) be a twisted coaction of H on A and v any unitary element in $A \otimes H$ with $(\text{id} \otimes \epsilon)(v) = 1$. Let

$$\rho_1 = \text{Ad}(v) \circ \rho, \quad u_1 = (v \otimes 1)(\rho \otimes \text{id})(v)u(\text{id} \otimes \Delta)(v^*).$$

Then (ρ_1, u_1) is a twisted coaction of H on A by easy computations.

Let $\text{Hom}(H^0, A)$ be the linear space of all linear maps from H^0 to A . By Sweedler [9, pp. 69–70] it becomes a unital $*$ -algebra which is also defined in [6, Sections 2 and 3]. In the same way as [6, Sections 2 and 3], we define a unital $*$ -algebra $\text{Hom}(H^0 \otimes H^0, A)$. As mentioned in Blattner, Cohen and Montgomery [2, pp. 163], there are an isomorphism ι of $A \otimes H$ onto $\text{Hom}(H^0, A)$ and an isomorphism j of $A \otimes H \otimes H$ onto $\text{Hom}(H^0 \otimes H^0, A)$ defined by

$$\begin{aligned}\iota(a \otimes h)(\phi) &= \phi(h)a, \\ j(a \otimes h \otimes l)(\phi, \psi) &= \phi(h)\psi(l)a\end{aligned}$$

for any $a \in A$, $h, l \in H$ and $\phi, \psi \in H^0$. For any $x \in A \otimes H$, $y \in A \otimes H \otimes H$, we denote $\iota(x)$, $j(y)$, by \hat{x} , \hat{y} , respectively.

For any weak coaction ρ of H on A , we can construct the weak action \cdot_ρ of H^0 on A as follows: For any $a \in A$ and $\phi \in H^0$

$$\phi \cdot_\rho a = \iota(\rho(a)) = \rho(a)\hat{\phi} = (\text{id} \otimes \phi)(\rho(a)).$$

If no confusion arises, we denote $\phi \cdot_\rho a$ by $\phi \cdot a$ for any $a \in A$ and $\phi \in H^0$. Furthermore, if (ρ, u) is a twisted coaction of H on A , \hat{u} is a unitary cocycle for the above weak action induced by ρ . We call the pair of the weak action and the unitary cocycle \hat{u} the *twisted action* of H^0 on A induced by (ρ, u) . By [6, Section 3], we can construct the twisted crossed product of A by H^0 which is denoted by $A \rtimes_{\rho, u} H^0$. Let $\hat{\rho}$ be the dual coaction of ρ , which is defined by for any $a \in A$, $\phi \in H^0$,

$$\hat{\rho}(a \rtimes_{\rho, u} \phi) = (a \rtimes_{\rho, u} \phi_{(1)}) \otimes \phi_{(2)},$$

where $a \rtimes_{\rho, u} \phi$ denotes the element in $A \rtimes_{\rho, u} H^0$ induced by $a \in A$ and $\phi \in H^0$. If no confusion arises, we denote it by $a \rtimes \phi$.

Let (ρ, u) be a twisted coaction of H on A and $A \rtimes_{\rho, u} H^0$ the twisted crossed product induced by (ρ, u) . Let E_1^ρ be the canonical conditional expectation from $A \rtimes_{\rho, u} H^0$ onto A defined by $E_1^\rho(a \rtimes \phi) = \phi(e)a$ for any $a \in A$, $\phi \in H^0$. We note that E_1^ρ is faithful by [6, Lemma 3.14]. Also, let \hat{V} be an element in $\text{Hom}(H^0, A \rtimes_{\rho, u} H^0)$ defined by $\hat{V}(\phi) = 1 \rtimes \phi$ for any $\phi \in H^0$. Let V be an element in $(A \rtimes_{\rho, u} H^0) \otimes H$ induced by \hat{V} . By [6, Lemma 3.12], we can see that V and \hat{V} are unitary elements in $(A \rtimes_{\rho, u} H^0) \otimes H$ and $\text{Hom}(H^0, A \rtimes_{\rho, u} H^0)$, respectively and that

$$u = (V \otimes 1)(\rho_H^{A \rtimes_{\rho, u} H^0} \otimes \text{id})(V)(\text{id} \otimes \Delta)(V^*).$$

Thus, for any $\phi, \psi \in H^0$

$$\begin{aligned}(1) \quad \hat{u}(\phi, \psi) &= \hat{V}(\phi_{(1)})\hat{V}^*(\psi_{(1)})\hat{V}^*(\phi_{(2)}\psi_{(2)}), \\ (2) \quad \hat{u}^*(\phi, \psi) &= \hat{V}(\phi_{(1)}\psi_{(1)})\hat{V}^*(\psi_{(2)})\hat{V}^*(\phi_{(2)}).\end{aligned}$$

Lemma 2.2. *For $i = 1, 2$ let (ρ_i, u_i) be a twisted coaction of H on A with $(\rho_1, u_1) \sim (\rho_2, u_2)$. Let $E_1^{\rho_i}$ be the canonical conditional expectation from $A \rtimes_{\rho_i, u_i} H^0$ onto A for $i = 1, 2$. Then there is an isomorphism Φ of $A \rtimes_{\rho_1, u_1} H^0$ onto $A \rtimes_{\rho_2, u_2} H^0$ satisfying that $\Phi(a) = a$ for any $a \in A$ and $E_1^{\rho_1} = E_1^{\rho_2} \circ \Phi$, where A is identified with $A \rtimes_{\rho_i, u_i} 1^0$ for $i = 1, 2$.*

Proof. Since $(\rho_1, u_1) \sim (\rho_2, u_2)$, there is a unitary element v in $A \otimes H$ satisfying that

$$\rho_2 = \text{Ad}(v) \circ \rho_1, \quad u_2 = (v \otimes 1)(\rho_1 \otimes \text{id})(v)u_1(\text{id} \otimes \Delta)(v^*).$$

Let Φ be a map from $A \rtimes_{\rho_1, u_1} H^0$ to $A \rtimes_{\rho_2, u_2} H^0$ defined by $\Phi(a \rtimes_{\rho_1, u_1} \phi) = a\hat{v}^*(\phi_{(1)}) \rtimes_{\rho_2, u_2} \phi_{(2)}$ for any $a \in A$, $\phi \in H^0$. Then by routine computations, Φ is a homomorphism of $A \rtimes_{\rho_1, u_1} H^0$ to $A \rtimes_{\rho_2, u_2} H^0$. Also, let Ψ be a map from $A \rtimes_{\rho_2, u_2} H^0$ to $A \rtimes_{\rho_1, u_1} H^0$ defined by for any $a \in A$, $\phi \in H^0$, $\Psi(a \rtimes_{\rho_2, u_2} \phi) = a\hat{v}(\phi_{(1)}) \rtimes_{\rho_1, u_1} \phi_{(2)}$. By routine computations, Ψ is also a homomorphism of $A \rtimes_{\rho_2, u_2} H^0$ to $A \rtimes_{\rho_1, u_1} H^0$ and $\Phi \circ \Psi = \text{id}$ and $\Psi \circ \Phi = \text{id}$. Therefore, we obtain the conclusion. \square

Let ρ be a coaction of H on A and A^ρ the fixed point C^* -subalgebra of A for ρ , that is,

$$A^\rho = \{a \in A \mid \rho(a) = a \otimes 1\}.$$

Let E^ρ be the canonical conditional expectation from A onto A^ρ defined by $E^\rho(a) = \tau \cdot_\rho a = (\text{id} \otimes \tau)(\rho(a))$ for any $a \in A$. We note that E^ρ is faithful by [10, Proposition 2.12].

Definition 2.3. We say that ρ is *saturated* if the action of H^0 on A induced by ρ is saturated in the sense of [10].

In Sections 4, 5 and 6 of [6], we suppose that the action of H on A is saturated. But, without saturation, all the statements in Sections 4 and 5 and Theorem 6.4 of [6] hold. Hence we obtain the following proposition.

Proposition 2.3. *Let ρ be a coaction of H on A such that $\widehat{\rho}(1 \rtimes \tau) \sim (1 \rtimes \tau) \otimes 1^0$ in $(A \rtimes_\rho H^0) \otimes H^0$. Then there are a twisted coaction (α, u) of H^0 on A^ρ and an isomorphism π of $A^\rho \rtimes_{\alpha, u} H$ onto A such that $E_1^\alpha = E^\rho \circ \pi$ and $\rho \circ \pi = (\pi \otimes \text{id}) \circ \widehat{\alpha}$.*

Corollary 2.4. *Let ρ be a coaction of H on A such that $\widehat{\rho}(1 \rtimes \tau) \sim (1 \rtimes \tau) \otimes 1^0$ in $(A \rtimes_\rho H^0) \otimes H^0$. Then ρ is saturated.*

Proof. Since the dual coaction of a twisted coaction is saturated, this is immediate by Proposition 2.3. \square

3. DUALITY

In this section we shall show the duality theorem for a twisted coaction of H^0 on A . It has already proved, but we shall present it in a useful form in this paper.

Let (ρ, u) be a twisted coaction of H^0 on A . Let Λ be the set of all triplets (i, j, k) , where $i, j = 1, 2, \dots, d_k$ and $k = 1, 2, \dots, K$ and $\sum_{k=1}^K d_k^2 = N$. For any $I = (i, j, k) \in \Lambda$, let W_I and V_I be elements in $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ defined by

$$\begin{aligned} W_I &= \sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k, \\ V_I &= (1 \rtimes_{\widehat{\rho}} \tau)(W_I \rtimes_{\widehat{\rho}} 1^0). \end{aligned}$$

Lemma 3.1. *With the above notations,*

$$V_I V_J^* = \begin{cases} 1 \rtimes_{\widehat{\rho}} \tau & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}$$

Proof. Let $I = (i, j, k)$ and $J = (s, t, r)$ be any elements in Λ . Then

$$\begin{aligned} V_I V_J^* &= (1 \rtimes_{\widehat{\rho}} \tau)(W_I \rtimes_{\widehat{\rho}} 1^0)(W_J^* \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau) \\ &= (1 \rtimes_{\widehat{\rho}} \tau)(W_I W_J^* \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau) = [\tau_{(1)} \cdot_{\widehat{\rho}} W_I W_J^*] \rtimes_{\widehat{\rho}} \tau_{(2)} \tau' \\ &= [\tau \cdot_{\widehat{\rho}} W_I W_J^*] \rtimes_{\widehat{\rho}} \tau = E_1^\rho(W_I W_J^*) \rtimes_{\widehat{\rho}} \tau, \end{aligned}$$

where $\tau' = \tau$. Here, by [6, Lemma 3.3 (1)] and [10, Theorem 2.2]

$$\begin{aligned}
W_I W_J^* &= (\sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k) (\sqrt{d_r} \rtimes_{\rho, u} w_{st}^r)^* \\
&= \sum_{t_1, t_2} \sqrt{d_k d_r} (1 \rtimes_{\rho, u} w_{ij}^k) (\widehat{u}(S(w_{t_2 t_1}^r), w_{st_2}^r)^* \rtimes_{\rho, u} w_{t_1 t}^{r*}) \\
&= \sum_{t_1, t_2, j_1, j_2, m} \sqrt{d_k d_r} [w_{ij_2}^k \cdot_{\rho, u} \widehat{u}(S(w_{t_2 t_1}^r), w_{st_2}^r)^*] \widehat{u}(w_{j_2 j_1}^k, w_{t_1 m}^{r*}) \rtimes_{\rho, u} w_{j_1 j}^k w_{mt}^{r*} \\
&= \sum_{t_1, t_2, j_1, j_2, m} \sqrt{d_k d_r} [w_{j_2 i}^k \cdot_{\rho, u} \widehat{u}(S(w_{t_2 t_1}^r), w_{st_2}^r)^*] \widehat{u}(w_{j_2 j_1}^k, w_{t_1 m}^{r*}) \rtimes_{\rho, u} w_{j_1 j}^k w_{mt}^{r*} \\
&= \sum_{t_1, t_2, t_3, j_1, j_2, j_3, m} \sqrt{d_k d_r} [\widehat{u}(w_{j_2 j_3}^k, S(w_{t_3 t_1}^r)) \widehat{u}(w_{j_3 i}^k, S(w_{t_2 t_3}^r), w_{st_2}^r)^*] \\
&\quad \times \widehat{u}(w_{j_2 j_1}^k, w_{t_1 m}^{r*}) \rtimes_{\rho, u} w_{j_1 j}^k w_{mt}^{r*} \\
&= \sum_{t_1, t_2, t_3, j_1, j_2, j_3, m} \sqrt{d_k d_r} \widehat{u}(w_{j_3 i}^k, S(w_{t_2 t_3}^r), w_{st_2}^r)^* \widehat{u}^*(w_{j_3 j_2}^k, w_{t_3 t_1}^{r*}) \widehat{u}(w_{j_2 j_1}^k, w_{t_1 m}^{r*}) \\
&\quad \rtimes_{\rho, u} w_{j_1 j}^k w_{mt}^{r*} \\
&= \sum_{t_2, t_3, j_3} \sqrt{d_k d_r} \widehat{u}(w_{j_3 i}^k, S(w_{t_2 t_3}^r), w_{st_2}^r)^* \rtimes_{\rho, u} w_{j_3 j}^k w_{t_3 t}^{r*}.
\end{aligned}$$

Thus, by [10, Theorem 2.2]

$$V_I V_J^* = \sum_{t_2, t_3, j_3} \sqrt{d_k d_r} \tau(w_{j_3 j}^k w_{t_3 t}^{r*}) \widehat{u}^*(w_{ij_3}^k w_{t_2 t_3}^{r*}, w_{t_2 s}^r) \rtimes_{\widehat{\rho}} \tau.$$

If $k \neq r$ or $j \neq t$, then $V_I V_J^* = 0$. We suppose that $k = r$ and $j = t$. Then

$$V_I V_J^* = \sum_{t_2, t_3} \widehat{u}^*(w_{it_3}^k, S(w_{t_3 t_2}^k), w_{t_2 s}^r) \rtimes_{\widehat{\rho}} \tau = \epsilon(w_{is}^k) \rtimes_{\widehat{\rho}} \tau = \delta_{is} \rtimes_{\widehat{\rho}} \tau,$$

where δ_{is} is the Kronecker delta. Therefore by [10, Theorem 2.2], we obtain the conclusion. \square

Let Ψ be a map from $M_N(A)$ to $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ defined by

$$\Psi([a_{IJ}]) = \sum_{I, J} V_I^* (a_{IJ} \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J$$

for any $[a_{IJ}] \in M_N(A)$. Clearly Ψ is a linear map.

Proposition 3.2. *The map Ψ is an isomorphism of $M_N(A)$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$.*

Proof. For any $[a_{IJ}], [b_{IJ}] \in M_N(A)$,

$$\begin{aligned}
\Psi([a_{IJ}]) \Psi([b_{IJ}]) &= \sum_{I, J, L} V_I^* (a_{IJ} \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) (1 \rtimes_{\widehat{\rho}} \tau) (b_{JL} \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_L \\
&= \sum_{I, J, L} V_I^* (1 \rtimes_{\widehat{\rho}} \tau) (a_{IJ} b_{JL} \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_L \\
&= \sum_{I, J, L} V_I^* (a_{IJ} b_{JL} \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_L = \Psi([a_{IJ}][b_{IJ}])
\end{aligned}$$

by Lemma 3.1. For any $[a_{IJ}] \in M_N(A)$,

$$\Psi([a_{IJ}])^* = \sum_{I, J} V_J^* (a_{IJ}^* \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_I = \Psi([a_{JI}^*]).$$

Hence Ψ is a homomorphism of $M_N(A)$ to $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$. Since $\widehat{\rho}$ is saturated, for any $z \in A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$, we can write that

$$z = \sum_{i=1}^n (x_i \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y_i \rtimes_{\widehat{\rho}} 1^0)$$

by [10, Proposition 4.5], where $x_i, y_i \in A \rtimes_{\rho,u} H$ for $i = 1, 2, \dots, n$. Thus, in order to prove that Ψ is surjective, it suffices to show that for any $x, y \in A \rtimes_{\rho,u} H$, there is an element $[a_{IJ}] \in M_N(A)$ such that $\Psi([a_{IJ}]) = (x \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y \rtimes_{\widehat{\rho}} 1^0)$. Since $\{(W_I^*, W_I)\}$ is a quasi-basis for E_ρ by [6, Proposition 3.18],

$$\begin{aligned} x &= \sum_I W_I^* E_1^\rho(W_I x) = \sum_I W_I^* (E_1^\rho(W_I x) \rtimes_{\rho,u} 1), \\ y &= \sum_I E_1^\rho(y W_I^*) W_I = \sum_I (E_1^\rho(y W_I^*) \rtimes_{\rho,u} 1) W_I. \end{aligned}$$

Hence

$$\begin{aligned} (x \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y \rtimes_{\widehat{\rho}} 1^0) &= \sum_{I,J} (W_I^* E_1^\rho(W_I x) \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(E_1^\rho(y W_J^*) W_J \rtimes_{\widehat{\rho}} 1^0) \\ &= \sum_{I,J} (W_I^* \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(E_1^\rho(W_I x) E_1^\rho(y W_J^*) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(W_J \rtimes_{\widehat{\rho}} 1^0) \\ &= \sum_{I,J} V_I^* (E_1^\rho(W_I x) E_1^\rho(y W_J^*) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J \\ &= \Psi([E_1^\rho(W_I x) E_1^\rho(y W_J^*)]_{I,J}). \end{aligned}$$

Next, we shall show that Ψ is injective. We suppose that for an element $[a_{IJ}] \in M_N(A)$, $\Psi([a_{IJ}]) = 0$. Then $\sum_{I,J} V_I^* (a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J = 0$. Thus for any $M, L \in \Lambda$,

$$0 = V_M \sum_{I,J} V_I^* (a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J V_L^* = a_{ML} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0$$

by Lemma 3.1. Hence $a_{ML} = 0$ for any $M, L \in \Lambda$. Therefore, Ψ is injective. \square

Since $V_I V_I^* = 1 \rtimes_{\widehat{\rho}} \tau$ for any $I \in \Lambda$ by Lemma 3.1, the set $\{V_I^* V_I\}_{I \in \Lambda}$ is a family of orthogonal projections in $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$. Let $P_I = V_I^* V_I$ for any $I \in \Lambda$. By Lemma 3.1 and Proposition 3.2,

$$1 = \Psi(1 \otimes I_N) = \sum_{I \in \Lambda} V_I^* V_I = \sum_{I \in \Lambda} P_I,$$

where I_N is the unit element in $M_N(\mathbf{C})$.

We recall that \widehat{V} is a unitary element in $\text{Hom}(H, A \rtimes_{\rho,u} H)$ defined by $\widehat{V}(h) = 1 \rtimes_{\rho,u} h$ for any $h \in H$. Let V be the unitary element in $(A \rtimes_{\rho,u} H) \otimes H^0$ induced by \widehat{V} . We regard $A \rtimes_{\rho,u} H$ as a C^* -subalgebra $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} 1^0$ of $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$. Thus we regard V as a unitary element in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$. For any $I \in \Lambda$, let

$$U_I = (V_I^* \otimes 1^0) V \widehat{\rho}(V_I) \in (A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0.$$

Then for any $I \in \Lambda$, $U_I U_I^* = P_I \otimes 1^0$ and $U_I^* U_I = \widehat{\rho}(P_I)$ since

$$\widehat{\rho}(1 \rtimes_{\widehat{\rho}} \tau) = V^*[(1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0] V$$

by [6, Proposition 3.19]. Let $U = \sum_{I \in \Lambda} U_I$. Then U is a unitary element in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$. Since (ρ, u) is a twisted coaction of H^0 on A , $(\rho \otimes \text{id}_{M_N(\mathbf{C})}, u \otimes I_N)$ is also a twisted coaction of H^0 on $M_N(A)$. Then by easy computations,

$$((\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi^{-1}, (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N))$$

is a twisted coaction of H^0 on $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$, where we identify $A \otimes M_N(\mathbf{C}) \otimes H^0 \otimes H^0$ with $A \otimes H^0 \otimes H^0 \otimes M_N(\mathbf{C})$.

Theorem 3.3. *Let A be a unital C^* -algebra and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let (ρ, u) be a twisted coaction of H^0 on A . Then there is an isomorphism Ψ of $M_N(A)$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ and a unitary element $U \in (A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ such that*

$$\begin{aligned} \text{Ad}(U) \circ \widehat{\rho} &= (\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi^{-1}, \\ (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N) &= (U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*). \end{aligned}$$

That is, $\widehat{\rho}$ is exterior equivalent to the twisted coaction

$$((\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi^{-1}, (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N)).$$

Proof. Let Ψ be the isomorphism of $M_N(\mathbf{C})$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ defined in Proposition 3.2 and let U be a unitary element in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ defined above. Let $[a_{IJ}]_{I,J \in \Lambda}$ be any element in $M_N(A)$. Then

$$\begin{aligned} (\text{Ad}(U) \circ \widehat{\rho})(\Psi([a_{IJ}])) &= U \widehat{\rho} \left(\sum_{I,J} V_I^* (a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J \right) U^* \\ &= \sum_{I,J} (V_I^* \otimes 1^0) V \widehat{\rho}(1 \rtimes_{\widehat{\rho}} \tau) \widehat{\rho}(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \widehat{\rho}(1 \rtimes_{\widehat{\rho}} \tau) V^* (V_J \otimes 1^0) \\ &= \sum_{I,J} (V_I^* \otimes 1^0) [(1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0] V ((a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \otimes 1^0) V^* [(1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0] (V_J \otimes 1^0) \end{aligned}$$

since $\widehat{\rho}(1 \rtimes_{\widehat{\rho}} \tau) = V^*[(1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0]V$ by [6, Proposition 3.19], where we identify A with $A \rtimes_{\rho,u} 1$ and $A \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0$. Hence

$$(\text{Ad}(U) \circ \widehat{\rho})(\Psi([a_{IJ}])) = \sum_{I,J} (V_I^* \otimes 1^0) \rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) (V_J \otimes 1^0)$$

since $\rho(a) = V(a \otimes 1^0)V^*$ for any $a \in A$ by [6, Lemma 3.12(1)]. On the other hand,

$$((\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}))([a_{IJ}]) = (\Psi \otimes \text{id}_{H^0})([\rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)]).$$

Since $\rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \in A \otimes H^0$, we can write that

$$\rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) = \sum_i (b_{IJi} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \otimes \phi_{IJi},$$

where $b_{IJi} \in A$ and $\phi_{IJi} \in H^0$ for any I, J, i . Hence

$$\begin{aligned} (\Psi \otimes \text{id}_{H^0})([\rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)]) &= (\Psi \otimes \text{id}_{H^0}) \left(\sum_i (b_{IJi} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \otimes \phi_{IJi} \right) \\ &= \sum_i V_I^* (b_{IJi} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J \otimes \phi_{IJi} \\ &= \sum_i (V_I^* \otimes 1^0) [(b_{IJi} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \otimes \phi_{IJi}] (V_J \otimes 1^0) \\ &= \sum_i (V_I^* \otimes 1^0) \rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) (V_J \otimes 1^0). \end{aligned}$$

Thus we obtain that

$$\text{Ad}(U) \circ \widehat{\rho} \circ \Psi = (\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}).$$

Next, we shall show that

$$(\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*).$$

Since $u \in A \otimes H^0 \otimes H^0$, we can write that $u = \sum_{i,j} a_{ij} \otimes \phi_i \otimes \psi_j$, where $a_{ij} \in A$ and $\phi_i, \psi_j \in H^0$ for any i, j . Thus for any $h, l \in H$

$$\begin{aligned} (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N) \widehat{}(h, l) &= \sum_{I,i,j} V_I^*(a_{ij} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_I \phi_i(h) \psi_j(l) \\ &= \sum_I V_I^*(\widehat{u}(h, l) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_I. \end{aligned}$$

On the other hand, by Lemma 3.1 and [6, Proposition 3.19]

$$\begin{aligned} &(U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*) \\ &= [\sum_I (V_I^* \otimes 1^0 \otimes 1^0)(V \otimes 1^0)(\widehat{\rho}(V_I) \otimes 1^0)] [\sum_J (\widehat{\rho}(V_J^*) \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(V) \\ &\quad \times ((\widehat{\rho} \otimes \text{id}) \circ \widehat{\rho})(V_J)] [\sum_L (\text{id} \otimes \Delta^0)(\widehat{\rho}(V_L^*))(\text{id} \otimes \Delta^0)(V^*)(V_L \otimes 1^0 \otimes 1^0)] \\ &= \sum_I (V_I^* \otimes 1^0 \otimes 1^0)(V \otimes 1^0)(\widehat{\rho}(1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(V) \\ &\quad \times ((\text{id} \otimes \Delta^0) \circ \widehat{\rho})(1 \rtimes_{\widehat{\rho}} \tau)(\text{id} \otimes \Delta^0)(V^*)(V_I \otimes 1^0 \otimes 1^0) \\ &= \sum_I (V_I^* \otimes 1^0 \otimes 1^0)((1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0 \otimes 1^0)(V \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(V) \\ &\quad \times (\text{id} \otimes \Delta^0)(V^*)((1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0 \otimes 1^0)(V_I \otimes 1^0 \otimes 1^0) \\ &= \sum_I (V_I^* \otimes 1^0 \otimes 1^0)(V \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(V)(\text{id} \otimes \Delta^0)(V^*)(V_I \otimes 1^0 \otimes 1^0). \end{aligned}$$

Thus for any $h, l \in H$,

$$\begin{aligned} &[(U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*)] \widehat{}(h, l) \\ &= \sum_I V_I^*[(V \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(V)(\text{id} \otimes \Delta^0)(V^*)] \widehat{}(h, l) V_I. \end{aligned}$$

Here for any $h, l \in H$

$$\begin{aligned} &[(V \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(V)(\text{id} \otimes \Delta^0)(V^*)] \widehat{}(h, l) = \widehat{V}(h_{(1)})[h_{(2)} \cdot_{\widehat{\rho}} \widehat{V}(l_{(1)})] \widehat{V}^*(h_{(3)} l_{(2)}) \\ &= \widehat{V}(h_{(1)})[h_{(2)} \cdot_{\widehat{\rho}} (1 \rtimes_{\rho,u} l_{(1)} \rtimes_{\widehat{\rho}} 1^0)] \widehat{V}^*(h_{(3)} l_{(2)}) \\ &= \widehat{V}(h_{(1)})(1 \rtimes_{\rho,u} l_{(1)} \rtimes_{\widehat{\rho}} 1^0) \widehat{V}^*(h_{(2)} l_{(2)}) \\ &= \widehat{V}(h_{(1)}) \widehat{V}(l_{(1)}) \widehat{V}^*(h_{(2)} l_{(2)}) = \widehat{u}(h, l) \end{aligned}$$

by [6, Lemma 3.12]. Thus

$$(V \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(V)(\text{id} \otimes \Delta^0)(V^*) = u.$$

Therefore

$$(\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*).$$

□

4. APPROXIMATELY REPRESENTABLE COACTIONS

For a unital C^* -algebra A , we set

$$\begin{aligned} c_0(A) &= \{(a_n) \in l^\infty(\mathbf{N}, A) \mid \lim_{n \rightarrow \infty} \|a_n\| = 0\}, \\ A^\infty &= l^\infty(\mathbf{N}, A)/c_0(A). \end{aligned}$$

We denote an element in A^∞ by the same symbol (a_n) in $l^\infty(\mathbf{N}, A)$. We identify A with the C^* -subalgebra of A^∞ consisting of the equivalence classes of constant sequences and set

$$A_\infty = A^\infty \cap A'.$$

For a weak coaction of H^0 on A , let ρ^∞ be the weak coaction of H^0 on A^∞ defined by $\rho^\infty((a_n)) = (\rho(a_n))$ for any $(a_n) \in A^\infty$. Hence for a twisted coaction (ρ, u) of H^0 on A , we can define the twisted coaction (ρ^∞, u) of H^0 on A^∞ . We have the following easy lemmas.

Lemma 4.1. *Let (ρ, u) be a twisted coaction of H^0 on A and (ρ^∞, u) the twisted coaction of H^0 on A^∞ induced by (ρ, u) . Then*

$$A^\infty \rtimes_{\rho^\infty, u} H \cong (A \rtimes_{\rho, u} H)^\infty$$

as C^* -algebras.

Proof. Let Φ be a map from $A^\infty \rtimes_{\rho^\infty, u} H$ to $(A \rtimes_{\rho, u} H)^\infty$ defined by $\Phi((a_n) \rtimes h) = (a_n \rtimes h)$ for any $(a_n) \in A^\infty$ and $h \in H$. For any $(a_n), (b_n) \in A^\infty$ with $(a_n) = (b_n)$ in A^∞ ,

$$\|a_n \rtimes h - b_n \rtimes h\| \leq \|a_n - b_n\| \|h\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Hence Φ is well-defined. Also, clearly Φ is linear. For $x \in A^\infty \rtimes_{\rho^\infty, u} H$, we suppose that $\Phi(x) = 0$. Then we can write that $x = \sum_i (x_{ni}) \rtimes h_i$, where $x_{ni} \in A$ and $\{h_i\}$ is a basis of H such that $\tau(h_i h_j) = 0$ and δ_{ij} is the Kronecker delta. Since $\Phi(x) = 0$, $\|\sum_i a_{ni} \rtimes_{\rho, u} h_i\| \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\|(\sum_i a_{ni} \rtimes_{\rho, u} h_i)(\sum_j a_{nj} \rtimes_{\rho, u} h_j)^*\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Also, by the proof of [6, Lemma 3.14]

$$E_1^\rho((\sum_i x_{ni} \rtimes_{\rho, u} h_i)(\sum_j x_{nj} \rtimes_{\rho, u} h_j)^*) = \sum_i x_{ni} x_{ni}^*.$$

Thus $\|\sum_i x_{ni} x_{ni}^*\| \rightarrow 0$ as $n \rightarrow \infty$. Hence for any i , $x_{ni} \rightarrow 0$ as $n \rightarrow \infty$. That is, $x = 0$. Thus Φ is injective. For any $x \in (A \rtimes_{\rho, u} H)^\infty$, we write $x = (x_n)$, $x_n = \sum_{i,j,k} x_{nij}^k \rtimes w_{ij}^k$, where $x_{nij}^k \in A$. Then $y = \sum_{i,j,k} (x_{nij}^k) \rtimes w_{ij}^k$ is an element in $A^\infty \rtimes_{\rho^\infty, u} H$ and $\Phi(y) = x$. Hence Φ is surjective. Furthermore, by routine computations, we see that Φ is a homomorphism of $A^\infty \rtimes_{\rho^\infty, u} H$ to $(A \rtimes_{\rho, u} H)^\infty$. Therefore, we obtain the conclusion. \square

By the isomorphism defined in the above lemma, we identify $A^\infty \rtimes_{\rho^\infty, u} H$ with $(A \rtimes_{\rho, u} H)^\infty$. Thus $\widehat{(\rho^\infty)} = (\widehat{\rho})^\infty$. We denote them by $\widehat{\rho}^\infty$.

Lemma 4.2. *Let ρ be a coaction of H^0 on A and ρ^∞ the coaction of H^0 on A^∞ induced by ρ . Then $(A^\infty)^{\rho^\infty} = (A^\rho)^\infty$.*

Proof. It is clear that $(A^\rho)^\infty \subset (A^\infty)^{\rho^\infty}$. We shall show that $(A^\rho)^\infty \supset (A^\infty)^{\rho^\infty}$. Let E^ρ and $(E)^\rho$ be the canonical conditional expectations from A and A^∞ onto A^ρ and $(A^\infty)^{\rho^\infty}$, respectively. Then $(A^\infty)^{\rho^\infty} = (E)^\rho(A^\infty)$ and $A^\rho = E^\rho(A)$. Let $(a_n)_n \in (A^\infty)^{\rho^\infty}$. We note that

$$\begin{aligned} (a_n)_n &= (E)^\rho((a_n)_n) = e \cdot_{\rho^\infty} (a_n)_n = (\text{id} \otimes e)(\rho^\infty((a_n)_n)) = (\text{id} \otimes e)((\rho(a_n))_n) \\ &= ((\text{id} \otimes e)(\rho(a_n)))_n = (E^\rho(a_n))_n. \end{aligned}$$

Hence $\|E^\rho(a_n) - a_n\| \rightarrow 0$ ($n \rightarrow \infty$). Let $b_n = E^\rho(a_n)$ for any $n \in \mathbf{N}$. Since $b_n \in A^\rho$, $(b_n) \in (A^\rho)^\infty$. Then $\|b_n - a_n\| = \|E^\rho(a_n) - a_n\| \rightarrow 0$ ($n \rightarrow \infty$). Thus $(b_n) = (a_n)$ in $(A^\rho)^\infty$. Therefore, $(a_n) \in (A^\rho)^\infty$. \square

Since $(A^\infty)^{\rho^\infty} = (A^\rho)^\infty$ by the above lemma, we can identify $(E)^{\rho^\infty}$ with $(E^\rho)^\infty$ the conditional expectation from A^∞ onto $(A^\rho)^\infty$. We denote them by E^{ρ^∞} .

Definition 4.1. Let (ρ, u) be a twisted coaction of H on A . We say that (ρ, u) is *approximately representable* if there is a unitary element $w \in A^\infty \otimes H$ satisfying the following conditions:

- (1) $\rho(a) = (\text{Ad}(w) \circ \rho_H^A)(a)$ for any $a \in A$,
- (2) $u = (w \otimes 1)(\rho_H^{A^\infty} \otimes \text{id})(w)(\text{id} \otimes \Delta)(w^*)$,
- (3) $u = (\rho^\infty \otimes \text{id})(w)(w \otimes 1)(\text{id} \otimes \Delta)(w^*)$.

Lemma 4.3. For $i = 1, 2$, let (ρ_i, u_i) be a twisted coaction of H on A . We suppose that (ρ_1, u_1) is exterior equivalent to (ρ_2, u_2) . Then (ρ_1, u_1) is approximately representable if and only if (ρ_2, u_2) is approximately representable.

Proof. Since (ρ_1, u_1) and (ρ_2, u_2) are exterior equivalent, there is a unitary element $v \in A \otimes H$ satisfying Conditions (1), (2) in Definition 2.2. We suppose that (ρ_1, u_1) is approximately representable. Then there is a unitary element $w_1 \in A^\infty \otimes H$ satisfying Conditions (1)-(3) in Definition 4.1 for (ρ_1, u_1) . Let $w_2 = vw_1$. Then by routine computations, we can see that w_2 is a unitary element in $A^\infty \otimes H$ satisfying Conditions (1)-(3) in Definition 4.1 for (ρ_2, u_2) . Therefore, we obtain the conclusion. \square

Lemma 4.4. Let (ρ, u) be a twisted coaction of H on A and let $(\rho \otimes \text{id}, u \otimes I_n)$ be the twisted coaction of H on $A \otimes M_n(\mathbf{C})$ induced by (ρ, u) , where we identify $A \otimes M_n(\mathbf{C}) \otimes H$ with $A \otimes H \otimes M_n(\mathbf{C})$. Then (ρ, u) is approximately representable if and only if $(\rho \otimes \text{id}, u \otimes I_n)$ is approximately representable.

Proof. We suppose that (ρ, u) is approximately representable. Then there is a unitary element $w \in A^\infty \otimes H$ satisfying Conditions (1)-(3) in Definition 4.1 for (ρ, u) . Let $W = w \otimes I_n$. By routine computations, we can see that W satisfies Conditions (1)-(3) in Definition 4.1 for $(\rho \otimes \text{id}, u \otimes I_n)$. Next, we suppose that $(\rho \otimes \text{id}, u \otimes I_n)$ is approximately representable. Then there is a unitary element $W \in A \otimes M_n(\mathbf{C}) \otimes H$ satisfying Conditions (1)-(3) in Definition 4.1 for $(\rho \otimes \text{id}, u \otimes I_n)$. Let f be a minimal projection in $M_n(\mathbf{C})$ and let $p_0 = 1_A \otimes f \otimes 1_H$. Let $w = p_0 W p_0$. Since $\rho \otimes \text{id}_{M_n(\mathbf{C})} = \text{Ad}(W) \circ \rho_H^{A \otimes M_n(\mathbf{C})}$ on $A \otimes M_n(\mathbf{C})$, $W p_0 = p_0 W$. By routine computations and identifying $A \otimes M_n(\mathbf{C}) \otimes H$ with $A \otimes H \otimes M_n(\mathbf{C})$, we can see that the element w satisfies Conditions (1)-(3) in Definition 4.1 for (ρ, u) . Therefore, we obtain the conclusion. \square

Proposition 4.5. Let (ρ, u) be a twisted coaction of H on A . Then (ρ, u) is approximately representable if and only if so is $\widehat{\widehat{\rho}}$.

Proof. This is immediate by Theorem 3.3 and Lemmas 4.3, 4.4. \square

In the rest of this section, we shall show that the approximate representability of coactions of finite dimensional C^* -Hopf algebras is an extension of the approximate representability of actions of finite groups in the sense of Izumi [4, Remark 3.7].

Let G be a finite group with the order n and α an action of G on A . We consider the coaction of $C(G)$ on A induced by the action α of G on A . We denote it by the same symbol α . That is,

$$\alpha : A \longrightarrow A \otimes C(G), \quad a \longmapsto \sum_{t \in G} \alpha_t(a) \otimes \delta_t$$

for any $a \in A$, where for any $t \in G$, δ_t is a projection in $C(G)$ defined by

$$\delta_t(s) = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases}.$$

Proposition 4.6. *With the above notations, the following conditions are equivalent:*

- (1) *the action α of G on A is approximately representable,*
- (2) *the coaction α of $C(G)$ on A is approximately representable.*

Proof. We suppose Condition (1). Then there is a unitary representation u of G in A^∞ such that

$$\begin{aligned}\alpha_t(a) &= u(t)au(t)^* \quad a \in A, t \in G, \\ \alpha_t^\infty(u(s)) &= u(tst^{-1}) \quad s, t \in G,\end{aligned}$$

where α^∞ is the automorphism of A^∞ induced by α . Let w be a unitary element in $A^\infty \otimes C(G)$ defined by $w = \sum_{t \in G} u(t) \otimes \delta_t$. Since u is a unitary representation of G in A^∞ , we obtain Condition (2) in Definition 4.1 for the coaction α . Also, by the above two conditions, we obtain Conditions (1) and (3) in Definition 4.1 for the coaction α . Next we suppose Condition (2). Then there is a unitary element $w \in A^\infty \otimes C(G)$ satisfying Condition (1)-(3) in Definition 4.1 for the coaction α . We can regard $A^\infty \otimes C(G)$ as the C^* -algebra of all A^∞ -valued functions on G . Hence there is the function from G to A^∞ corresponding to w . We denote it by u . Since w is a unitary element in $A^\infty \otimes C(G)$, $u(t)$ is a unitary element in A^∞ for any $t \in G$. By easy computations, Condition (2) in Definition 4.1 for the coaction α implies that u is a unitary representation of G in A^∞ . Also, Conditions (1) and (3) in Definition 4.1 for the coaction α imply that

$$\begin{aligned}\alpha_t(a) &= u(t)au(t)^* \quad a \in A, t \in G, \\ \alpha_t^\infty(u(s)) &= u(tst^{-1}) \quad s, t \in G.\end{aligned}$$

Therefore, we obtain the conclusion. \square

5. COACTIONS WITH THE ROHLIN PROPERTY

In this section, we shall introduce the Rohlin property for coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra.

Definition 5.1. Let (ρ, u) be a twisted coaction of H^0 on A . We say that (ρ, u) has the *Rohlin property* if the dual coaction $\widehat{\rho}$ of H on $A \rtimes_{\rho, u} H$ is approximately representable.

First, we shall begin with the following easy propositions.

Proposition 5.1. *Let ρ be a coaction of H^0 on A with the Rohlin property. Then ρ is saturated.*

Proof. This is immediate by Corollary 2.4. \square

Proposition 5.2. *Let (ρ, u) be a twisted coaction of H^0 on A . Then (ρ, u) has the Rohlin property if and only if so does $\widehat{\widehat{\rho}}$.*

Proof. This is immediate by Proposition 4.5. \square

Let (ρ, u) be a twisted coaction of H^0 on A with the Rohlin property. Then there is a unitary element $w \in (A^\infty \rtimes_{\rho^\infty, u} H) \otimes H$ satisfying that

- (5, 1) $\widehat{\rho}(x) = (\text{Ad}(w) \circ \rho_H^{A \rtimes_{\rho^\infty, u} H})(x)$ for any $x \in A \rtimes_{\rho, u} H$,
- (5, 2) $(w \otimes 1)(\rho_H^{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \text{id}_H)(w) = (\text{id}_{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \Delta)(w)$,
- (5, 3) $(\widehat{\rho}^\infty \otimes \text{id}_H)(w)(w \otimes 1) = (\text{id}_{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \Delta)(w)$.

Let \widehat{w} be the element in $\text{Hom}(H^0, A^\infty \rtimes_{\rho^\infty, u} H)$ induced by w .

Lemma 5.3. *With the above notations, \widehat{w} is a homomorphism of H^0 to $(A^\infty \rtimes_{\rho^\infty, u} H) \cap A'$ satisfying the following conditions:*

- (1) $\widehat{w}(1^0) = 1_{A^\infty}$,
- (2) *the element $\widehat{w}(\tau)$ is a projection in A_∞ ,*
- (3) $\widehat{w}(\tau)x\widehat{w}(\tau) = E_1^\rho(x)\widehat{w}(\tau)$ for any $x \in A \rtimes_{\rho, u} H$.

Proof. By Equation (5, 2), $\widehat{w} \in \text{Alg}(H^0, A^\infty \rtimes_{\rho^\infty} H)$. Furthermore, by [2, Lemma 1.16], $\widehat{w}^* = \widehat{w} \circ S^0$. Thus for any $\phi \in H^0$, $\widehat{w}(\phi)^* = \widehat{w}^*(S^0(\phi^*)) = \widehat{w}(\phi^*)$. Hence \widehat{w} is a homomorphism of H^0 to $A^\infty \rtimes_{\rho^\infty} H$. Next we shall show that $\widehat{w}(\phi)(a \rtimes 1) = (a \rtimes 1)\widehat{w}(\phi)$ for any $a \in A$. By Equation (5, 1), for any $a \in A$,

$$(a \rtimes 1) \otimes 1 = w[(a \rtimes 1) \otimes 1]w^*.$$

Thus $[(a \rtimes 1) \otimes 1]w = w[(a \rtimes 1) \otimes 1]$. Hence for any $\phi \in H^0$

$$(a \rtimes 1)\widehat{w}(\phi) = \widehat{w}(\phi)(a \rtimes 1).$$

Hence \widehat{w} is a homomorphism of H^0 to $(A^\infty \rtimes_{\rho^\infty, u} H) \cap A'$. Also, by Equation (5, 2),

$$\begin{aligned} & (\text{id}_{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \epsilon \otimes \text{id}_H)((w \otimes 1)(\rho_H^{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \text{id}_H)(w)) \\ &= (\text{id}_{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \epsilon \otimes \text{id}_H)((\text{id}_{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \Delta)(w)). \end{aligned}$$

Thus $[(\text{id}_{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \epsilon)(w) \otimes 1]w = w$. Since w is a unitary element in $(A^\infty \rtimes_{\rho^\infty, u} H) \otimes H$, $(\text{id}_{A^\infty \rtimes_{\rho^\infty, u} H} \otimes \epsilon)(w) = 1$, that is, $\widehat{w}(1^0) = 1$. Furthermore, since τ is a projection in H^0 and \widehat{w} is a homomorphism of H^0 to $A^\infty \rtimes_{\rho^\infty} H$, $\widehat{w}(\tau)$ is a projection. Also, by Equation (5, 3), for any $\phi \in H^0$

$$\begin{aligned} \phi \cdot_{\widehat{\rho}^\infty} \widehat{w}(\tau) &= \widehat{w}(\phi_{(1)}\tau)\widehat{w}^*(\phi_{(2)}) = \widehat{w}(\tau)\widehat{w}^*(\phi) = \widehat{w}(\tau)\widehat{w}(S^0(\phi^*))^* = \widehat{w}(\tau)\widehat{w}(S^0(\phi)) \\ &= \epsilon^0(\phi)\widehat{w}(\tau). \end{aligned}$$

Hence by [6, Lemma 3.17], $\widehat{w}(\tau) \in A^\infty \cap A' = A_\infty$. Finally, we shall show that $\widehat{w}(\tau)x\widehat{w}(\tau) = E_1^\rho(x)\widehat{w}(\tau)$ for any $x \in A \rtimes_{\rho, u} H$. For any $a \in A$, $h \in H$, $\widehat{\rho}(a \rtimes h) = w[(a \rtimes h) \otimes 1]w^*$. Thus

$$(a \rtimes h_{(1)})\tau(h_{(2)}) = \widehat{w}(\tau_{(1)})(a \rtimes h)\widehat{w}^*(\tau_{(2)}).$$

That is, $\tau(h)(a \rtimes 1) = \widehat{w}(\tau_{(1)})(a \rtimes h)\widehat{w}^*(\tau_{(2)})$. Since $E_1^\rho(a \rtimes h) = \tau(h)(a \rtimes 1)$ and $\widehat{w}^* = \widehat{w} \circ S^0$,

$$\begin{aligned} E_1^\rho(a \rtimes h)\widehat{w}(\tau) &= \tau(h)(a \rtimes 1)\widehat{w}(\tau) = \widehat{w}(\tau_{(1)})(a \rtimes h)\widehat{w}^*(\tau_{(2)})\widehat{w}(\tau) \\ &= \widehat{w}(\tau_{(1)})(a \rtimes h)\widehat{w}(\tau)\epsilon^0(\tau_{(2)}) = \widehat{w}(\tau)(a \rtimes h)\widehat{w}(\tau). \end{aligned}$$

Thus we obtain the last condition. \square

Proposition 5.4. *For $i = 1, 2$, let (ρ_i, u_i) be a twisted coaction of H^0 on A with $(\rho_1, u_1) \sim (\rho_2, u_2)$. Then (ρ_1, u_1) has the Rohlin property if and only if so does (ρ_2, u_2) .*

Proof. Since $(\rho_1, u_1) \sim (\rho_2, u_2)$, there is a unitary element $v \in A \otimes H^0$ satisfying that

$$\rho_2 = \text{Ad}(v) \circ \rho_1 \quad u_2 = (v \otimes 1^0)(\rho_1 \otimes \text{id})(v)u_1(\text{id} \otimes \Delta^0)(v^*).$$

Then there is an isomorphism Φ of $A \rtimes_{\rho_1, u_1} H$ onto $A \rtimes_{\rho_2, u_2} H$ defined by

$$\Phi(a \rtimes_{\rho_1, u_1} h) = a\widehat{v}^*(h_{(1)}) \rtimes_{\rho_2, u_2} h_{(2)}.$$

By easy computations, we can see that the following conditions hold:

- (1) $\widehat{\rho}_2 \circ \Phi = (\Phi \otimes \text{id}_H) \circ \widehat{\rho}_1$,
- (2) $\rho_H^{A \rtimes_{\rho_2, u_2} H} \circ \Phi = (\Phi \otimes \text{id}_H) \circ \rho_H^{A \rtimes_{\rho_1, u_1} H}$,
- (3) $(\text{id}_{A \rtimes_{\rho_2, u_2} H} \otimes \Delta) \circ (\Phi \otimes \text{id}_H) = (\Phi \otimes \text{id}_H \otimes \text{id}_H) \circ (\text{id}_{A \rtimes_{\rho_1, u_1} H} \otimes \Delta)$

Let Φ^∞ be the isomorphism of $A^\infty \rtimes_{\rho_1, u_1} H$ onto $A^\infty \rtimes_{\rho_2, u_2} H$ induced by Φ . We suppose that (ρ_1, u_1) has the Rohlin property and let w_1 be a unitary element in $(A \rtimes_{\rho_1, u_1} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for the coaction $\hat{\rho}_1$. Let $w_2 = (\Phi^\infty \otimes \text{id}_H)(w_1)$. By Conditions (1)-(3), we can see that w_2 satisfies Equations (5, 1)-(5, 3) for the coaction $\hat{\rho}_2$. Therefore we obtain the conclusion. \square

Lemma 5.5. *For $i = 1, 2$, let (ρ_i, u_i) be a twisted coaction of H^0 on A with $(\rho_1, u_1) \sim (\rho_2, u_2)$. We suppose that (ρ_i, u_i) has the Rohlin property for $i = 1, 2$. Let w_i be as in the above proof for $i = 1, 2$. Then $\widehat{w}_1(\tau) = \widehat{w}_2(\tau)$.*

Proof. Let $w_1 = \sum_{i,j} (a_{ij} \rtimes_{\rho_1, u_1} h_i) \otimes l_j$, where $a_{ij} \in A^\infty$. Then

$$w_2 = \sum_{i,j} (a_{ij} \widehat{v}^*(h_{i(1)}) \rtimes_{\rho_2, u_2} h_{i(2)}) \otimes l_j,$$

where v is a unitary element in $A \otimes H^0$ defined in the above proof. Thus

$$\begin{aligned} \widehat{w}_2(\tau) &= \sum_{i,j} (a_{ij} \widehat{v}^*(h_{i(1)}) \rtimes_{\rho_2, u_2} h_{i(2)}) \tau(l_j) \\ &= \Phi \left(\sum_{i,j} (a_{ij} \rtimes_{\rho_1, u_1} h_i) \tau(l_j) \right) = \Phi(\widehat{w}_1(\tau)), \end{aligned}$$

where Φ is the isomorphism of $A \rtimes_{\rho_1, u_1} H$ onto $A \rtimes_{\rho_2, u_2} H$ defined in the above proof. On the other hand, since $\widehat{w}_1(\tau) \in A_\infty \subset A^\infty$ by Lemma 5.3, $\widehat{w}_2(\tau) = \Phi(\widehat{w}_1(\tau)) = \widehat{w}_1(\tau)$. \square

Let (ρ, u) be a twisted coaction of H on A with the Rohlin property. Let w be a unitary element in $(A^\infty \rtimes_{\rho^\infty, u} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for $\hat{\rho}$.

Lemma 5.6. *With the above notations, $e \cdot \widehat{w}(\tau) = \frac{1}{N}$.*

Proof. We note that $\hat{\rho}(1 \rtimes h) = w[(1 \rtimes h) \otimes 1]w^*$ for any $h \in H$. Since $\widehat{w}^* = \widehat{w} \circ S^0$, we see that for any $h \in H$, $(1 \rtimes h)\widehat{w}(S^0(\tau)) = \widehat{w}(S^0(\tau_{(1)}))(1 \rtimes h_{(1)})\tau_{(2)}(h_{(2)})$. Hence for any $h \in H$, $\widehat{V}(h)\widehat{w}(\tau) = \widehat{w}(S^0(\tau_{(1)}))\widehat{V}(h_{(1)})\tau_{(2)}(h_{(2)})$. Then

$$\begin{aligned} e \cdot \widehat{w}(\tau) &= \sum_{i,k} \frac{d_k}{N} [w_{ii}^k \cdot \widehat{w}(\tau)] = \sum_{i,j,k} \frac{d_k}{N} \widehat{V}(w_{ij}^k)(\widehat{w}(\tau) \rtimes 1) \widehat{V}^*(w_{ji}^k) \\ &= \sum_{i,j,k,j_1} \frac{d_k}{N} \widehat{w}(S^0(\tau_{(1)})) \widehat{V}(w_{ij_1}^k) \tau_{(2)}(w_{j_1 j}^k) \widehat{V}^*(w_{ji}^k) \\ &= \sum_{i,j,k,j_1} \frac{d_k}{N} \widehat{w}(S^0(\tau_{(1)}))(1 \rtimes w_{ij_1}^k)(1 \rtimes w_{ij}^k)^* \tau_{(2)}(w_{j_1 j}^k) \\ &= \sum_{i,j,k,j_1,j_2,j_3} \frac{d_k}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{j_1 j}^k)(1 \rtimes w_{ij_1}^k)(\widehat{w}(S(w_{j_2 j_3}^k), w_{ij_2}^k)^* \rtimes w_{j_3 j}^{k*}) \\ &= \sum_{i,j,k,j_1,j_2,j_3,j_4,j_5,s} \frac{d_k}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{j_1 j}^k) [w_{ij_4}^k \cdot \widehat{w}(S(w_{j_2 j_3}^k), w_{ij_2}^k)^*] \widehat{w}(w_{j_4 j_5}^k, w_{j_3 s}^{k*}) \\ &\quad \rtimes w_{j_5 j_1}^k w_{sj}^{k*} \end{aligned}$$

since $w_{ij}^{k*} = S(w_{ji}^k)$ for any i, j, k by [10, Theorem 2.2 2]. Since $e \cdot \widehat{w}(\tau) \in A^\infty$, $E_1^{\rho^\infty}(e \cdot \widehat{w}(\tau)) = e \cdot \widehat{w}(\tau)$. Thus since $\tau(w_{ij}^k w_{st}^{r*}) = \frac{1}{d_k} \delta_{kr} \delta_{is} \delta_{jt}$ by [10, Theorem 2.2,

2],

$$\begin{aligned}
e \cdot \widehat{w}(\tau) &= \sum_{i,j,k,j_2,j_3,j_4,s} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) [w_{ij_4} \cdot \widehat{u}(S(w_{j_2j_3}^k), w_{ij_2}^k)^*] \\
&\quad \times \widehat{u}(w_{j_4s}^k, w_{j_3s}^{k*}) \\
&= \sum_{i,j,k,j_2,j_3,j_4,s} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) [w_{ij_4} \cdot \widehat{u}(S(w_{j_2j_3}^k), w_{ij_2}^k)^*] \widehat{u}(w_{j_4s}^k, w_{j_3s}^{k*}).
\end{aligned}$$

Furthermore by [6, Lemma 3.3(1)],

$$\begin{aligned}
e \cdot \widehat{w}(\tau) &= \sum_{i,j,k,j_2,j_3,j_4,s,t,r} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) (\widehat{u}(w_{j_4t}^k, S(w_{rj_3}^k))) \\
&\quad \times \widehat{u}(w_{ti}^k S(w_{j_2r}^k), w_{ij_2}^k)^* \widehat{u}(w_{j_4s}^k, w_{j_3s}^{k*}) \\
&= \sum_{i,j,k,j_2,j_3,j_4,s,t,r} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) \widehat{u}^*(S(w_{ti}^{k*}) w_{j_2r}^{k*}, S(w_{ij_2}^{k*})) \\
&\quad \times \widehat{u}^*(S(w_{j_4t}^{k*}), w_{rj_3}^{k*}) \widehat{u}(w_{j_4s}^k, w_{j_3s}^{k*}) \\
&= \sum_{i,j,k,j_2,j_3,j_4,s,t,r} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) \widehat{u}^*(w_{it}^k S(w_{rj_2}^k), w_{j_2i}^k) \\
&\quad \times \widehat{u}^*(w_{tj_4}^k, w_{rj_3}^{k*}) \widehat{u}(w_{j_4s}^k, w_{j_3s}^{k*}) \\
&= \sum_{i,j,k,j_2,s} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) \widehat{u}^*(w_{is}^k S(w_{sj_2}^k), w_{j_2i}^k) \\
&= \sum_{i,j,k} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) \epsilon(w_{ii}^k).
\end{aligned}$$

Since $e = \sum_{ik} \frac{d_k}{N} w_{ii}^k$ and $\epsilon(w_{ij}^k) = \delta_{ij}$ for any k by [10, Theorem 2.2, 2],

$$e \cdot \widehat{w}(\tau) = \sum_{j,k} \frac{d_k}{N} \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(w_{jj}^k) = \widehat{w}(S^0(\tau_{(1)})) \tau_{(2)}(e) = \frac{1}{N}$$

by Lemma 5.3(1). Therefore, we obtain the conclusion. \square

By Lemmas 5.3(2) and 5.6, we can see that if ρ is a coaction of H^0 on A with the Rohlin property, then there is a projection $p \in A_\infty$ such that $e \cdot p = \frac{1}{N}$. We shall show the inverse direction with the assumption that ρ is saturated. Let ρ be a saturated coaction of H^0 on A . We suppose that there is a projection $p \in A_\infty$ such that $e \cdot p = \frac{1}{N}$.

Lemma 5.7. *With the above notations and assumptions, for any $x \in A \rtimes H$, $(p \rtimes 1)x(p \rtimes 1) = E_1^\rho(x)(p \rtimes 1)$.*

Proof. Let $q = N(p \rtimes 1)(1 \rtimes e)(p \rtimes 1)$. Then q is a projection in $A^\infty \rtimes_{\rho^\infty} H$. Indeed, $q^* = q$. Also, $q^2 = N^2(p \rtimes 1)([e \cdot p] \rtimes e)(p \rtimes 1) = q$ by the assumption. Furthermore, $E_1^{\rho^\infty}(q) = p = E_1^{\rho^\infty}(p \rtimes 1)$. Since $q \leq p$ and $E_1^{\rho^\infty}$ is faithful, we obtain that $p = q$. That is, $p = N(p \rtimes 1)(1 \rtimes e)(p \rtimes 1)$. For any $a, b \in A$,

$$\begin{aligned}
(p \rtimes 1)(a \rtimes 1)(1 \rtimes e)(b \rtimes 1)(p \rtimes 1) &= (a \rtimes 1)(p \rtimes 1)(1 \rtimes e)(p \rtimes 1)(b \rtimes 1) \\
&= \frac{1}{N}(ab \rtimes 1)(p \rtimes 1).
\end{aligned}$$

Since ρ is saturated, $A(1 \rtimes e)A = A \rtimes_\rho H$. Hence we obtain the conclusion. \square

By Watatani [11, Proposition 2.2.7 and Lemma 2.2.9] and Lemma 5.7, we can see that there is a homomorphism π of $A \rtimes_{\rho} H \rtimes_{\widehat{\rho}} H^0$ to $A^{\infty} \rtimes_{\rho^{\infty}} H$ such that

$$\pi((x \rtimes 1^0)(1_A \rtimes 1_H \rtimes \tau)(y \rtimes 1^0)) = x(p \rtimes 1)y$$

for any $x, y \in A \rtimes_{\rho} H$. The restriction of π to $1_{A \rtimes_{\rho} H} \rtimes H^0$ is a homomorphism of H^0 to $A^{\infty} \rtimes_{\rho^{\infty}} H$. Thus there is an element $w \in (A^{\infty} \rtimes_{\rho^{\infty}} H) \otimes H$ such that \widehat{w} is the above restriction of π to H^0 . Let $\{(u_i, u_i^*)\}$ be a quasi-basis of E_1^{ρ} .

Lemma 5.8. *With the above notations and assumptions, for any $\phi \in H^0$, $\widehat{w}(\phi) = \sum_j [\phi \cdot_{\widehat{\rho}} u_j](p \rtimes 1)u_j^*$.*

Proof. We note that $\tau \cdot x = E_1^{\rho}(x)$ for any $x \in A \rtimes_{\rho} H$. Since $\sum_i (u_i \rtimes 1^0)(1 \rtimes \tau)(u_i^* \rtimes 1^0) = 1$,

$$\begin{aligned} 1 \rtimes \phi &= \sum_i (1 \rtimes \phi)(u_i \rtimes 1^0)(1 \rtimes \tau)(u_i^* \rtimes 1^0) = \sum_i ([\phi_{(1)} \cdot u_i] \rtimes \phi_{(2)} \tau)(u_i^* \rtimes 1^0) \\ &= \sum_i ([\phi \cdot u_i] \rtimes 1^0)(1 \rtimes \tau)(u_i^* \rtimes 1^0). \end{aligned}$$

Hence we obtain the conclusion by the definition of \widehat{w} . \square

Lemma 5.9. *With the above notations, $\widehat{w}(1^0) = 1_A$.*

Proof. By [6, Proposition 3.18], $\{((\sqrt{d_k} \rtimes w_{ij}^k)^*, \sqrt{d_k} \rtimes w_{ij}^k)\}$ is a quasi-basis of E_1^{ρ} . Hence by Lemma 5.8,

$$\begin{aligned} \widehat{w}(1^0) &= \sum_{i,j,k} d_k (1 \rtimes w_{ij}^{k*})(p \rtimes w_{ij}^k) = \sum_{i,j,k,t} d_k [w_{it}^{k*} \cdot p] \rtimes w_{tj}^{k*} w_{ij}^k \\ &= \sum_{i,j,k,t} [w_{it}^{k*} \cdot p] \rtimes S(w_{jt}^k) w_{ij}^k = \sum_{i,k,t} [S(w_{ti}^k) \cdot p] \rtimes \epsilon(w_{it}^k) \\ &= \sum_{i,k} [S(w_{ii}^k) \cdot p] \rtimes 1 = N[e \cdot p] = 1. \end{aligned}$$

\square

Lemma 5.10. *With the above notations, the element w is a unitary element in $(A^{\infty} \rtimes_{\rho^{\infty}} H) \otimes H$ satisfying Equations (5, 1) – (5, 3).*

Proof. Since \widehat{w} is a homomorphism of H^0 to $A^{\infty} \rtimes_{\rho^{\infty}} H$, the element w satisfies Equation (5, 2). Also, for any $\phi \in H^0$

$$(\widehat{w}\widehat{w}^*)(\phi) = \widehat{w}(\phi_{(1)})\widehat{w}(S^0(\phi_{(2)})^*)^* = \widehat{w}(\phi_{(1)})S^0(\phi_{(2)}) = \epsilon^0(\phi)\widehat{w}(1^0) = \epsilon^0(\phi)$$

by Lemma 5.9. Similarly $(\widehat{w}^*\widehat{w})(\phi) = \epsilon^0(\phi)$. Hence w is a unitary element in $(A^{\infty} \rtimes_{\rho^{\infty}} H) \otimes H$. Let $\{(u_i, u_i^*)\}$ be a quasi-basis of E_1^{ρ} . By Lemmas 5.7 and 5.8

for any $\phi, \psi \in H^0$,

$$\begin{aligned}
[\phi_{(1)} \cdot \widehat{\rho} \widehat{w}(\psi)] \widehat{w}(\phi_{(2)}) &= \sum_{i,j} [\phi_{(1)} \cdot ([\psi \cdot u_j](p \rtimes 1) u_j^*)] [\phi_{(2)} \cdot u_i](p \rtimes 1) u_i^* \\
&= \sum_{i,j} [\phi \cdot ([\psi \cdot u_j](p \rtimes 1) u_j^* u_i)] (p \rtimes 1) u_i^* \\
&= \sum_{i,j} [\phi \cdot ([\psi \cdot u_j](p \rtimes 1) u_j^* u_i (p \rtimes 1))] u_i^* \\
&= \sum_{i,j} [\phi \cdot ([\psi \cdot u_j] E_1^p(u_j^* u_i)(p \rtimes 1))] u_i^* \\
&= \sum_{i,j} [\phi \cdot ([\psi \cdot (u_j E_1^p(u_j^* u_i))](p \rtimes 1))] u_i^* \\
&= \sum_i [\phi \cdot ([\psi \cdot u_i](p \rtimes 1))] u_i^* = \sum_i [(\phi \psi) \cdot u_i](p \rtimes 1) u_i^* = \widehat{w}(\phi \psi).
\end{aligned}$$

Thus the element w satisfies Equation (5, 3). Finally, for any $a \in A$, $h \in H$ and $\phi \in H^0$,

$$\begin{aligned}
\widehat{w}(\phi_{(1)})(a \rtimes h) \widehat{w}^*(\phi_{(2)}) &= \sum_{i,j} [\phi_{(1)} \cdot u_j](p \rtimes 1) u_j^*(a \rtimes h) [S^0(\phi_{(2)}) \cdot u_i](p \rtimes 1) u_i^* \\
&= \sum_{i,j} [\phi_{(1)} \cdot u_j] E_1^p(u_j^*(a \rtimes h) [S^0(\phi_{(2)}) \cdot u_i]) (p \rtimes 1) u_i^* \\
&= \sum_{i,j} [\phi_{(1)} \cdot (u_j E_1^p(u_j^*(a \rtimes h) [S^0(\phi_{(2)}) \cdot u_i]))] (p \rtimes 1) u_i^* \\
&= \sum_i [\phi_{(1)} \cdot ((a \rtimes h) [S^0(\phi_{(2)}) \cdot u_i])] (p \rtimes 1) u_i^* \\
&= \sum_i (a \rtimes h_{(1)}) \phi_{(1)}(h_{(2)}) [\phi_{(2)} \cdot [S^0(\phi_{(3)}) \cdot u_i]] (p \rtimes 1) u_i^* \\
&= \sum_i (a \rtimes h_{(1)}) \phi(h_{(2)}) u_i (p \rtimes 1) u_i^* \\
&= (a \rtimes h_{(1)}) \phi(h_{(2)}) \widehat{w}(1^0) = (a \rtimes h_{(1)}) \phi(h_{(2)})
\end{aligned}$$

by Lemmas 5.8 and 5.9. Hence w satisfies Equation (5, 1). Therefore we obtain the conclusion. \square

Theorem 5.11. *Let ρ be a coaction of a finite dimensional C^* -Hopf algebra H on a unital C^* -algebra A . If ρ is saturated, then the following conditions are equivalent:*

- (1) *the coaction ρ has the Rohlin property,*
- (2) *there is a projection p in A_∞ such that $e \cdot_\rho p = \frac{1}{N}$, where $N = \dim H$.*

Proof. This is immediate by Lemmas 5.6 and 5.10. \square

In the rest of this section, we shall show that the Rohlin property of coactions of a finite dimensional C^* -Hopf algebra is an extension of the Rohlin property of a finite group in the sense of [4, Remark 3.7]. Let G and α be as in the end of Section 4.

Proposition 5.12. *With the above notations, the following conditions are equivalent:*

- (1) *the action α of G on A has the Rohlin property,*
- (2) *the coaction α of $C(G)$ on A has the Rohlin property.*

Proof. We suppose Condition (1). Then there is a partition of unity $\{e_t\}_{t \in G}$ consisting of projections in A_∞ satisfying that $\alpha_t^\infty(e_s) = e_{ts}$ for any $t, s \in G$. By easy

computations, e_e is a projection in A_∞ such that $\tau \cdot e_e = \frac{1}{n}$, where τ is the Haar trace on $C(G)$. Since the coaction α of $C(G)$ on A is saturated, by Theorem 5.11 the coaction α has the Rohlin property. Next, we suppose Condition (2). Then there is a projection $p \in A_\infty$ such that $\tau \cdot p = \frac{1}{n}$ by Theorem 5.11. Hence

$$(\text{id} \otimes \tau) \left(\sum_{t \in G} \alpha_t^\infty(p) \otimes \delta_t \right) = \frac{1}{n}.$$

Thus, we see that $\sum_{t \in G} \alpha_t^\infty(p) = 1$ by the definition of τ . Let $e_t = \alpha_t^\infty(p)$ for any $t \in G$. Then clearly, $\{e_t\}_{t \in G}$ is a partition of unity consisting of projections in A_∞ satisfying that $\alpha_t^\infty(e_s) = e_{ts}$. \square

6. ANOTHER EQUIVALENT CONDITION

In this section, we shall give another condition which is equivalent to the Rohlin property.

Let (ρ, u) be a twisted coaction of H^0 on A . We suppose that (ρ, u) has the Rohlin property. Then there is a unitary element $w \in (A^\infty \rtimes_{\rho^\infty, u} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for (ρ, u) . Let \widehat{w} be the unitary element in $\text{Hom}(H^0, A^\infty \rtimes_{\rho^\infty, u} H)$ induced by $w \in (A \rtimes_{\rho, u} H) \otimes H$. By Lemma 5.3, $\widehat{w}(\tau)$ is a projection in A_∞ . By Theorem 3.3 there are an isomorphism Ψ of $M_N(A)$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ and a unitary element U in $(A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ such that

$$\begin{aligned} \text{Ad}(U) \circ \widehat{\rho} &= (\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbb{C})}) \circ \Psi^{-1}, \\ (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N) &= (U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*). \end{aligned}$$

Let $\sigma = (\Psi \otimes \text{id}) \circ (\rho \otimes \text{id}_{M_N(\mathbb{C})}) \circ \Psi^{-1}$ and $W = (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N)$. Then (σ, W) is a twisted coaction of H^0 on $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ which is exterior equivalent to $\widehat{\rho}$. Let $\widehat{\Psi}$ be the isomorphism of $M_N(A) \rtimes_{\rho \otimes \text{id}, u \otimes I_N} H$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0 \rtimes_{\sigma, W} H$ induced by Ψ , which is defined by $\widehat{\Psi}(x \rtimes_{\rho \otimes \text{id}, u \otimes I_N} h) = \Psi(x) \rtimes_{\sigma, W} h$ for any $x \in M_N(A)$, $h \in H$. Let $\widehat{\Psi}^\infty$ be the isomorphism of $M_N(A^\infty) \rtimes_{\rho^\infty \otimes \text{id}, u \otimes I_N} H$ onto $A^\infty \rtimes_{\rho^\infty, u} H \rtimes_{\widehat{\rho}^\infty} H^0 \rtimes_{\sigma^\infty, W} H$ induced by $\widehat{\Psi}$. By easy computations, (σ, W) has the Rohlin property and the unitary element $(\widehat{\Psi}^\infty \otimes \text{id}_H)(w \otimes I_N)$ is in $(A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0 \rtimes_{\sigma, W} H) \otimes H$ and satisfies Equations (5, 1)-(5, 3) for the twisted coaction (σ, W) . Let $z = (\widehat{\Psi}^\infty \otimes \text{id}_H)(w \otimes I_N)$. Then

$$\begin{aligned} \widehat{z}(\tau) &= ((\text{id} \otimes \tau) \circ (\widehat{\Psi}^\infty \otimes \text{id}_H))(w \otimes I_N) = \widehat{\Psi}^\infty((\text{id} \otimes \tau)(w \otimes I_N)) \\ &= \widehat{\Psi}^\infty(\widehat{w}(\tau) \otimes I_N) = \Psi^\infty(\widehat{w}(\tau) \otimes I_N). \end{aligned}$$

Lemma 6.1. *With the above notations and assumptions,*

$$\sum_{i,j,k} (\sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k)^* \widehat{w}(\tau) (\sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k) = 1.$$

Proof. By Proposition 5.2, $\widehat{\rho}$ has the Rohlin property. Let z_1 be a unitary element in $(A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0 \rtimes_{\widehat{\rho}} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for $\widehat{\rho}$. Then by Lemmas 5.5 and 5.6, $e \cdot_{\widehat{\rho}} \widehat{z}_1(\tau) = e \cdot_{\widehat{\rho}} \widehat{z}(\tau) = \frac{1}{N}$. Since $\widehat{z}(\tau) = \Psi^\infty(\widehat{w}(\tau) \otimes I_N)$,

$$\frac{1}{N} = e \cdot_{\widehat{\rho}} \Psi^\infty(\widehat{w}(\tau) \otimes I_N) = \sum_I [e \cdot_{\widehat{\rho}} V_I^*(\widehat{w}(\tau) \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_I].$$

Since $V_I = (1 \rtimes_{\widehat{\rho}} \tau)(W_I \rtimes_{\widehat{\rho}} 1^0)$,

$$\frac{1}{N} = \sum_I (W_I^* \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau(1)) \tau(2)(e_{(1)})(\widehat{w}(\tau) \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau'(1)) \tau'(2)(e_{(2)})(W_I \rtimes_{\widehat{\rho}} 1^0),$$

where $\tau' = \tau$. Furthermore,

$$\begin{aligned}
\frac{1}{N} &= \sum_I (W_I^* \rtimes_{\widehat{\rho}} 1^0) (1 \rtimes_{\widehat{\rho}} \tau_{(1)}) (\widehat{w}(\tau) \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) (1 \rtimes_{\widehat{\rho}} \tau'_{(1)}) (W_I \rtimes_{\widehat{\rho}} 1^0) (\tau_{(2)} \tau'_{(2)}) (e) \\
&= \sum_I (W_I^* \rtimes_{\widehat{\rho}} 1^0) (\widehat{w}(\tau) \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} \tau_{(1)} \tau'_{(1)}) (W_I \rtimes_{\widehat{\rho}} 1^0) (\tau_{(2)} \tau'_{(2)}) (e) \\
&= \sum_I (W_I^* \rtimes_{\widehat{\rho}} 1^0) (\widehat{w}(\tau) \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) (W_I \rtimes_{\widehat{\rho}} 1^0) (\tau \tau') (e) \\
&= \frac{1}{N} \sum_I (W_I^* \rtimes_{\widehat{\rho}} 1^0) (\widehat{w}(\tau) \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) (W_I \rtimes_{\widehat{\rho}} 1^0) \\
&= \frac{1}{N} \sum_I W_I^* (\widehat{w}(\tau) \rtimes_{\rho, u} 1) W_I.
\end{aligned}$$

Therefore we obtain the conclusion. \square

Next, we shall show the inverse direction of Lemma 6.1.

Lemma 6.2. *Let (ρ, u) be a twisted coaction of H^0 on A . We suppose that there is a projection $p \in A_\infty$ such that*

$$\sum_{i,j,k} (\sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k)^* (p \rtimes_{\rho, u} 1) (\sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k) = 1.$$

Then (ρ, u) has the Rohlin property.

Proof. Let Ψ be the isomorphism of $M_N(A)$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ defined in Theorem 3.3. Let $q = \Psi^\infty(p \otimes I_N)$. Then q is a projection in $(A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0)_\infty$ since $p \otimes I_N \in M_N(A)_\infty$. In the same way as in the proof of Lemma 6.1,

$$e \cdot_{\widehat{\rho}} q = e \cdot_{\widehat{\rho}} \Psi^\infty(p \otimes I_N) = \frac{1}{N} \sum_I W_I^* (p \rtimes_{\rho, u} 1) W_I = \frac{1}{N}.$$

Hence by Theorem 5.11, $\widehat{\rho}$ has the Rohlin property since $\widehat{\rho}$ is saturated by Jeong and Park [5, Theorem 3.3] and [6, Proposition 3.18]. Therefore (ρ, u) has the Rohlin property by Proposition 5.4. \square

Theorem 6.3. *Let (ρ, u) be a twisted coaction of a finite dimensional C^* -Hopf algebra H^0 on a unital C^* -algebra A . Let $\{w_{ij}^k\}$ be a system of comatrix units of H . Then the following conditions are equivalent:*

- (1) *the twisted coaction (ρ, u) has the Rohlin property,*
- (2) *there is a projection $p \in A_\infty$ such that $\sum_{i,j,k} (\sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k)^* p (\sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k) = 1$.*

Proof. This is immediate by Lemmas 6.1 and 6.2. \square

Corollary 6.4. *Let ρ be a coaction of H^0 on A . Then the following conditions are equivalent:*

- (1) *the coaction ρ has the Rohlin property,*
- (2) *there is a projection $p \in A_\infty$ such that $e \cdot_{\rho^\infty} p = \frac{1}{N}$.*

Proof. (1) implies (2): This is immediate by Lemma 5.6.

(2) implies (1): By Theorem 6.3, it suffices to show that (2) implies that

$$\sum_{i,j,k} (\sqrt{d_k} \rtimes_{\rho} w_{ij}^k)^* p (\sqrt{d_k} \rtimes_{\rho} w_{ij}^k) = 1.$$

Since ρ is a coaction of H^0 on A ,

$$\begin{aligned}
\sum_{i,j,k} (\sqrt{d_k} \rtimes_{\rho} w_{ij}^k)^* p(\sqrt{d_k} \rtimes_{\rho} w_{ij}^k) &= \sum_{i,j,k} d_k (1 \rtimes_{\rho} w_{ij}^{k*}) p(1 \rtimes_{\rho} w_{ij}^k) \\
&= \sum_{i,j,k} d_k (1 \rtimes_{\rho} S(w_{ji}^k)) p(1 \rtimes_{\rho} w_{ij}^k) = N \sum_{i,j,k} \frac{d_k}{N} \widehat{V}(S(w_{ji}^k)) p \widehat{V}^*(S(w_{ij}^k)) \\
&= N \sum_{i,k} \frac{d_k}{N} [S(w_{ii}^k) \cdot_{\rho^\infty} p] = N[S(e) \cdot_{\rho^\infty} p] = N[e \cdot_{\rho^\infty} p] = 1.
\end{aligned}$$

Therefore we obtain the conclusion. \square

7. EXAMPLE

In this section, we shall give an example of an approximately representable coaction of a finite dimensional C^* -Hopf algebra on a UHF-algebra which has also the Rohlin property.

We note that the comultiplication Δ^0 of H^0 can be regarded as a coaction of H^0 on a C^* -algebra H^0 . Hence we can consider the crossed product $H^0 \rtimes_{\Delta^0} H$, which is isomorphic to $M_N(\mathbf{C})$. Let $A = H^0 \rtimes_{\Delta^0} H$. Let $A_n = \otimes_1^n A$, the n -times tensor product of A , for any $n \in \mathbf{N}$. In the usual way, we regard A_n as a C^* -subalgebra of A_{n+1} , that is, for any $a \in A_n$, the map $\iota_n : a \mapsto a \otimes (1^0 \rtimes_{\Delta^0} 1)$ is regarded as the inclusion of A_n into A_{n+1} . Let B be the inductive limit C^* -algebra of $\{(A_n, \iota_n)\}$. Then B can be regarded as a UHF-algebra of type N^∞ . Let \widehat{V} be a unitary element in $\text{Hom}(H, A)$ defined by $\widehat{V}(h) = 1^0 \rtimes_{\Delta^0} h$ for any $h \in H$ and let V be the unitary element in $A \otimes H^0$ induced by \widehat{V} . Recall that $\{v_{ij}^k\}$ and $\{w_{ij}^k\}$ are systems of matrix units and comatrix units of H , respectively. Also, let $\{\phi_{ij}^k\}$ and $\{\omega_{ij}^k\}$ be systems of matrix units and comatrix units of H^0 , respectively. Let

$$v_1 = V = \sum_{i,j,k} (1^0 \rtimes_{\Delta^0} w_{ij}^k) \otimes \phi_{ij}^k = \sum_{i,j,k} (1^0 \rtimes_{\Delta^0} v_{ij}^k) \otimes \omega_{ij}^k.$$

For any $n \in \mathbf{N}$ with $n \geq 2$, let

$$\begin{aligned}
v_n &= (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes V \\
&= \sum_{i,j,k} (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes \widehat{V}(w_{ij}^k) \otimes \phi_{ij}^k \\
&= \sum_{i,j,k} (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes \widehat{V}(v_{ij}^k) \otimes \omega_{ij}^k.
\end{aligned}$$

Let $u_n = v_1 v_2 \cdots v_n \in A_n \otimes H^0$ for any $n \in \mathbf{N}$. Then u_n is a unitary element in $A_n \otimes H^0$ for any $n \in \mathbf{N}$.

Lemma 7.1. *With the above notations,*

$$\begin{aligned}
u_n &= \sum \widehat{V}(w_{i_1 j_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1} j}^k) \otimes \phi_{ij}^k \\
&= \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \widehat{V}(v_{j_1 j_2}^{k_2}) \otimes \cdots \otimes \widehat{V}(v_{j_{n-1} j}^{k_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{j_{n-1} j}^{k_n},
\end{aligned}$$

where the above summations are taken under all indices.

Proof. It is clear that the second equation holds. We show the first equation by the induction. We assume that

$$u_n = \sum \widehat{V}(w_{i_1 j_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1} j}^k) \otimes \phi_{ij}^k,$$

where the summation is taken under all indices. Then

$$\begin{aligned} u_{n+1} &= u_n v_{n+1} = \sum \widehat{V}(w_{ij_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1} j}^k) \otimes \widehat{V}(w_{st}^r) \otimes \phi_{ij}^k \phi_{st}^r \\ &= \sum \widehat{V}(w_{ij_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1} j}^k) \otimes \widehat{V}(w_{jt}^k) \otimes \phi_{it}^k, \end{aligned}$$

where the summations are taken under all indices. Therefore, we obtain the conclusion. \square

For any $n \in \mathbb{N}$, let $\rho_n = \text{Ad}(u_n) \circ \rho_{H^0}^{A_n}$, that is, for any $a \in A_n$,

$$\rho_n(a) = u_n(a \otimes 1^0)u_n^*.$$

Lemma 7.2. *With the above notations, ρ_n is a coaction of H^0 on A_n .*

Proof. We have only to show that

$$(u_n \otimes 1^0)(\rho_{H^0}^{A_n} \otimes \text{id})(u_n) = (\text{id} \otimes \Delta^0)(u_n).$$

By Lemma 7.1, we can write that

$$u_n = \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \widehat{V}(v_{i_2 j_2}^{k_2}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n},$$

where the summation is taken under all indices. Hence

$$\begin{aligned} u_n \otimes 1^0 &= \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n} \otimes 1^0, \\ (\rho_{H^0}^{A_n} \otimes \text{id})(u_n) &= \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \otimes 1^0 \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n}, \end{aligned}$$

where the summations are taken under all indices. Thus since \widehat{V} is a C^* -homomorphism of H to A ,

$$\begin{aligned} & (u_n \otimes 1^0)(\rho_{H^0}^{A_n} \otimes \text{id})(u_n) \\ &= \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \widehat{V}(v_{s_1 t_1}^{r_1}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \widehat{V}(v_{s_n t_n}^{r_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n} \otimes \omega_{s_1 t_1}^{r_1} \cdots \omega_{s_n t_n}^{r_n} \\ &= \sum \widehat{V}(v_{i_1 t_1}^{k_1}) \otimes \cdots \otimes \widehat{V}(v_{i_n t_n}^{k_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n} \otimes \omega_{j_1 t_1}^{k_1} \cdots \omega_{j_n t_n}^{k_n} \\ &= (\text{id} \otimes \Delta^0)(u_n), \end{aligned}$$

where the summations are taken under all indices. Therefore we obtain the conclusion. \square

Lemma 7.3. *With the above notations, $(\iota_n \otimes \text{id}) \circ \rho_n = \rho_{n+1} \circ \iota_n$ for any $n \in \mathbb{N}$.*

Proof. In this proof, the summations are taken under all indices. Let a be any element in A_n . Then by Lemma 7.1

$$\begin{aligned} \rho_n(a) &= u_n(a \otimes 1^0)u_n^* \\ &= \sum (\widehat{V}(w_{ij_1}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1} j}^k)) a (\widehat{V}(w_{st_1}^r)^* \otimes \cdots \otimes \widehat{V}(w_{t_{n-1} t}^r)^*) \otimes \phi_{ij}^k \phi_{ts}^r \\ &= \sum (\widehat{V}(w_{ij_1}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1} j}^k)) a (\widehat{V}(w_{st_1}^k)^* \otimes \cdots \otimes \widehat{V}(w_{t_{n-1} j}^k)^*) \otimes \phi_{is}^k. \end{aligned}$$

Hence

$$\begin{aligned} & ((\iota_n \otimes \text{id}) \circ \rho_n)(a) \\ &= \sum (\widehat{V}(w_{ij_1}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1} j}^k)) a (\widehat{V}(w_{st_1}^k)^* \otimes \cdots \otimes \widehat{V}(w_{t_{n-1} j}^k)^*) \otimes (1^0 \rtimes_{\Delta^0} 1) \\ & \quad \otimes \phi_{is}^k. \end{aligned}$$

On the other hand,

$$\begin{aligned}
(\rho_{n+1} \circ \iota_n)(a) &= \rho_{n+1}(a \otimes (1^0 \rtimes_{\Delta^0} 1)) \\
&= \sum (\widehat{V}(w_{ij_1}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1}j_n}^k)) a (\widehat{V}(w_{st_1}^k)^* \otimes \cdots \otimes \widehat{V}(w_{t_{n-1}t_n}^k)^*) \otimes \widehat{V}(w_{j_n j}^k w_{t_n j}^{k*}) \\
&\quad \otimes \phi_{is}^k \\
&= \sum (\widehat{V}(w_{ij_1}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1}j_n}^k)) a (\widehat{V}(w_{st_1}^k)^* \otimes \cdots \otimes \widehat{V}(w_{t_{n-1}t_n}^k)^*) \\
&\quad \otimes (1 \rtimes_{\Delta^0} 1) \epsilon(w_{j_n t_n}^k) \otimes \phi_{is}^k \\
&= ((\iota_n \otimes \text{id}) \circ \rho_n)(a)
\end{aligned}$$

since $w_{t_n j}^{k*} = S(w_{j t_n}^k)$. Therefore, we obtain the conclusion. \square

By Lemma 7.3, the inductive limit of $\{(\rho_n, \iota_n)\}$ is a homomorphism of B to $B \otimes H^0$. Furthermore, by Lemma 7.2, it is a coaction of H^0 on B . We denote it by ρ .

Proposition 7.4. *With the above notations, ρ is approximately representable.*

Proof. Let u be a unitary element in $B^\infty \otimes H^0$ defined by $u = (u_n)$, where A_n is regarded as C^* -subalgebra of B for any $n \in \mathbf{N}$. We can easily show that ρ and u hold the following conditions:

- (1) $\rho(x) = (\text{Ad}(u) \circ \rho_{H^0}^B)(x)$ for any $x \in B$,
- (2) $(u \otimes 1^0)(\rho_{H^0}^B \otimes \text{id})(u) = (\text{id} \otimes \Delta^0)(u)$,
- (3) $(\rho^\infty \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u)$.

Therefore, we obtain the conclusion. \square

Proposition 7.5. *With the above notations, ρ has the Rohlin property.*

Proof. By Corollary 6.4, it suffices to show that there is a projection $p \in B_\infty$ such that $e \cdot_{\rho^\infty} p = \frac{1}{N}$. For any $n \in \mathbf{N}$, let

$$p_n = (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes (\tau \rtimes_{\Delta^0} 1) \in A'_{n-1} \cap A_n.$$

Also, let $p = (p_n)$. Then clearly p is a projection in B_∞ . In order to show that $e \cdot_{\rho^\infty} p = \frac{1}{N}$, we have only to show that $e \cdot_{\rho_n} p_n = \frac{1}{N}$ for any $n \in \mathbf{N}$. We note that

$$\begin{aligned}
&u_n(p_n \otimes 1^0)u_n^* \\
&= \sum \widehat{V}(w_{ij_1}^k S(w_{t_1 s}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \\
&\quad \otimes \widehat{V}(w_{j_{n-1} j}^k (\tau \rtimes_{\Delta^0} 1) \widehat{V}(S(w_{j t_{n-1}}^k)) \otimes \phi_{is}^k,
\end{aligned}$$

where the summation is taken under all indices. Hence since $e = \sum_{f,q} \frac{d_f}{N} w_{qq}^f$,

$$\begin{aligned}
& e \cdot_{\rho_n} p_n \\
&= \sum \frac{d_f}{N} \widehat{V}(w_{i_{j_1}}^k S(w_{t_1 s}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \\
&\otimes \widehat{V}(w_{j_{n-1} j}^k (\tau \rtimes_{\Delta^0} 1) \widehat{V}(S(w_{j t_{n-1}}^k)) \phi_{is}^k(w_{qq}^f)) \\
&= \sum \frac{d_k}{N} \widehat{V}(w_{i_{j_1}}^k S(w_{t_1 i}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \\
&\otimes \widehat{V}(w_{j_{n-1} j}^k (\tau \rtimes_{\Delta^0} 1) \widehat{V}(S(w_{j t_{n-1}}^k)) \\
&= \sum \frac{d_k}{N} \widehat{V}(\epsilon(w_{t_1 j_1}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \\
&\otimes \widehat{V}(w_{j_{n-1} j}^k (\tau \rtimes_{\Delta^0} 1) \widehat{V}(S(w_{j t_{n-1}}^k)) \\
&= \sum \frac{d_k}{N} (1^0 \rtimes_{\Delta^0} 1) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 j_1}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \\
&\otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \otimes \widehat{V}(w_{j_{n-1} j}^k (\tau \rtimes_{\Delta^0} 1) \widehat{V}(S(w_{j t_{n-1}}^k)),
\end{aligned}$$

where the summations are taken under all indices. Doing this in the same way as in the above for $n-1$ times, we can obtain that

$$\begin{aligned}
e \cdot_{\rho_n} p_n &= \sum \frac{d_k}{N} (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes \widehat{V}(w_{j_{n-1} j}^k (\tau \rtimes_{\Delta^0} 1) \widehat{V}(S(w_{j t_{n-1}}^k)) \\
&= \sum \frac{d_k}{N} (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes ([w_{j_{n-1} j_{n-1}} \cdot_{\Delta^0} \tau] \rtimes_{\Delta^0} 1) \\
&= (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes ([e \cdot_{\Delta^0} \tau] \rtimes_{\Delta^0} 1) \\
&= (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes \tau_{(1)} \tau_{(2)}(e) (1^0 \rtimes_{\Delta^0} 1) \\
&= \frac{1}{N} (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1),
\end{aligned}$$

where the summations are taken under all indices. Therefore, we obtain the conclusion. \square

8. 1-COHOMOLOGY VANISHING THEOREM

Let ρ be a coaction of H^0 on A with the Rohlin property. In this section, we shall show that for any coaction σ of H^0 on A which is exterior equivalent to ρ , there is a unitary element $x \in A \otimes H^0$ such that $\sigma = \text{Ad}(x \otimes 1^0) \circ \rho \circ \text{Ad}(x^*)$.

Let ρ and σ be as above. Since ρ and σ are exterior equivalent, there is a unitary element $v \in A \otimes H^0$ satisfying the following conditions.

- (8, 1) $\sigma = \text{Ad}(v) \circ \rho$,
- (8, 2) $(v \otimes 1^0)(\rho \otimes \text{id}_{H^0})(v) = (\text{id} \otimes \Delta^0)(v)$.

Since ρ has the Rohlin property, there is a unitary element w in $(A \rtimes_{\sigma} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for $\widehat{\rho}$. By Proposition 5.4, σ has also the Rohlin property. Hence there is a unitary element $w_1 \in (A \rtimes_{\sigma} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for $\widehat{\sigma}$. By Lemma 5.5, $\widehat{w_1}(\tau) = \widehat{w}(\tau)$.

Let $x = N(\text{id} \otimes e)(v \rho^{\infty}(\widehat{w}(\tau))) = N\widehat{v}(e_{(1)})[e_{(2)} \cdot_{\rho^{\infty}} \widehat{w}(\tau)]$.

Lemma 8.1. *With the above notations, the element x is a unitary element in A^{∞} such that $\rho^{\infty}(x) = v^*(x \otimes 1^0)$.*

Proof. Let $f = e$. Then by Lemmas 5.3 and 5.5

$$\begin{aligned}
xx^* &= N^2 \widehat{v}(e_{(1)})[e_{(2)} \cdot_{\rho^\infty} \widehat{w}(\tau)][f_{(2)} \cdot_{\rho^\infty} \widehat{w}(\tau)]^* \widehat{v}(f_{(1)})^* \\
&= N^2 \widehat{v}(e_{(1)})[e_{(2)} \cdot_{\rho^\infty} \widehat{w}(\tau)][S(f_{(2)})^* \cdot_{\rho^\infty} \widehat{w}(\tau)] \widehat{v}(f_{(1)})^* \\
&= N^2 \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)}))(1 \rtimes_\rho S(f_{(3)}^*)) \widehat{w}(\tau)(1 \rtimes_\rho f_{(2)}^*) \widehat{v}(f_{(1)})^* \\
&= N^2 \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau) \tau(e_{(3)} f_{(3)}^*)(1 \rtimes_\rho f_{(2)}^*) \widehat{v}(f_{(1)})^* \\
&= N^2 \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)}) e_{(4)} f_{(2)}^*) \tau(e_{(5)} f_{(3)}^*) \widehat{v}(f_{(1)})^* \\
&= N^2 \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)})) \tau(e_{(4)} f_{(2)}^*) \widehat{v}(f_{(1)})^* \\
&= N^2 \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)})) \widehat{v}^*(S(f_{(1)}))^* \tau(e_{(4)} f_{(2)}^*) \\
&= N^2 \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)})) \widehat{v}^*(S(S(e_{(4)}) e_{(5)} f_{(1)}^*)) \tau(e_{(6)} f_{(2)}^*) \\
&= N^2 \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)})) \widehat{v}^*(e_{(4)}) \tau(e_{(5)} f) \\
&= N \widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)}) \widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)})) \widehat{v}^*(e_{(4)}) \\
&= N(\text{id} \otimes e)(v \rho^\infty(\widehat{w}(\tau)) v^*) \\
&= N[e \cdot_{\sigma^\infty} \widehat{w}_1(\tau)] = 1.
\end{aligned}$$

Let $y = N(\text{id} \otimes e)(v^* \sigma^\infty(\widehat{w}_1(\tau))) = N \widehat{v}^*(e_{(1)})[e_{(2)} \cdot_{\sigma^\infty} \widehat{w}_1(\tau)]$. Then by the above discussions, $yy^* = 1$. On the other hand, by Lemmas 5.5 and 5.6

$$\begin{aligned}
y^* &= N[S(e_{(2)}^*) \cdot_{\sigma^\infty} \widehat{w}(\tau)] \widehat{v}(S(e_{(1)}^*)) = N \widehat{v}(S(e_{(4)}^*)) [S(e_{(3)}^*) \cdot_{\rho^\infty} \widehat{w}(\tau)] \widehat{v}^*(S(e_{(2)}^*)) \widehat{v}(S(e_{(1)}^*)) \\
&= N \widehat{v}(S(e_{(2)}^*)) [S(e_{(1)}^*) \cdot_{\rho^\infty} \widehat{w}(\tau)] = N(\text{id} \otimes S(e)^*)(v \rho^\infty(\widehat{w}(\tau))) = x.
\end{aligned}$$

Thus $x^*x = 1$. Hence x is a unitary element in A^∞ . Finally, we shall show that $\rho^\infty(x) = v^*(x \otimes 1^0)$. Noting that $(v \otimes 1^0)(\rho \otimes \text{id})(v) = (\text{id} \otimes \Delta^0)(v)$,

$$\begin{aligned}
\rho^\infty(x) &= N \rho^\infty((\text{id} \otimes e)(v \rho^\infty(\widehat{w}(\tau)))) \\
&= N((\text{id} \otimes \text{id}_{H^0} \otimes e) \circ (\rho^\infty \otimes \text{id}_{H^0}))(v \rho^\infty(\widehat{w}(\tau))) \\
&= N(\text{id} \otimes \text{id}_{H^0} \otimes e)((\rho^\infty \otimes \text{id}_{H^0})(v)((\rho^\infty \otimes \text{id}_{H^0}) \circ \rho^\infty)(\widehat{w}(\tau))) \\
&= N(\text{id} \otimes \text{id}_{H^0} \otimes e)((v^* \otimes 1^0)(\text{id} \otimes \Delta^0)(v)((\text{id} \otimes \Delta^0) \circ \rho^\infty)(\widehat{w}(\tau))) \\
&= N v^*(\text{id} \otimes \text{id}_{H^0} \otimes e)((\text{id} \otimes \Delta^0)(v \rho^\infty(\widehat{w}(\tau)))) \\
&= N v^*(\text{id} \otimes e)(v \rho^\infty(\widehat{w}(\tau))) \otimes 1^0 \\
&= v^*(x \otimes 1^0).
\end{aligned}$$

□

Lemma 8.2. *With the above notations, for any $\epsilon > 0$ there is a unitary element x_0 in A such that*

$$\|v - (x_0 \otimes 1)\rho(x_0^*)\| < \epsilon.$$

Proof. By Lemma 8.1, there is a unitary element $x \in A^\infty$ such that $v = (x \otimes 1^0)\rho^\infty(x^*)$. Since x is a unitary element in A^∞ , for any $\epsilon > 0$, there is a unitary element $x_0 \in A$ such that $\|v - (x_0 \otimes 1)\rho(x_0^*)\| < \epsilon$. □

Theorem 8.3. *Let ρ and σ be coactions of H^0 on A which are exterior equivalent. We suppose that ρ has the Rohlin property. Then there is a unitary element $x \in A$ such that*

$$\sigma = \text{Ad}(x \otimes 1^0) \circ \rho \circ \text{Ad}(x^*).$$

Proof. Let v be a unitary element in $A \otimes H^0$ satisfying Equations (8, 1) and (8, 2). By Lemma 8.2, there is a unitary element $x_0 \in A$ such that

$$\|v - (x_0 \otimes 1)\rho(x_0^*)\| < 1.$$

Let

$$\rho_1 = \text{Ad}(x_0 \otimes 1) \circ \rho \circ \text{Ad}(x_0^*) = \text{Ad}((x_0 \otimes 1^0)\rho(x_0^*)) \circ \rho.$$

Let $v_1 = (x_0 \otimes 1^0)\rho(x_0^*)$. Then ρ_1 is a coaction of H^0 on A . Also, $\sigma = \text{Ad}(vv_1^*) \circ \rho_1$.

Let $v_2 = vv_1^*$. Then v_2 is a unitary element in $A \otimes H^0$ with

$$\|v_2 - 1\| = \|v - v_1\| = \|v - (x_0 \otimes 1^0)\rho(x_0^*)\| < 1.$$

Furthermore, since $v_1 = (x_0 \otimes 1^0)\rho(x_0^*)$,

$$\begin{aligned} (v_2 \otimes 1^0)(\rho_1 \otimes \text{id})(v_2) &= (v_2 \otimes 1^0)(v_1 \otimes 1^0)(\rho \otimes \text{id})(v_2)(v_1^* \otimes 1^0) \\ &= (v \otimes 1^0)(\rho \otimes \text{id})(v_2)(v_1^* \otimes 1^0) \\ &= (v \otimes 1^0)(\rho \otimes \text{id})(v)(\rho \otimes \text{id})(v_1^*)(v_1^* \otimes 1^0) \\ &= (\text{id} \otimes \Delta^0)(v)(\rho \otimes \text{id})(v_1^*)(v_1^* \otimes 1^0) \\ &= (\text{id} \otimes \Delta^0)(v_2)(\text{id} \otimes \Delta^0)(v_1)(\rho \otimes \text{id})(v_1^*)(v_1^* \otimes 1^0) \\ &= (\text{id} \otimes \Delta^0)(v_2)(x_0 \otimes 1^0 \otimes 1^0)((\text{id} \otimes \Delta^0) \circ \rho)(x_0^*) \\ &\quad \times ((\rho \otimes \text{id}) \circ \rho)(x_0)(\rho(x_0^*) \otimes 1^0)(\rho(x_0) \otimes 1^0)(x_0^* \otimes 1^0 \otimes 1^0) \\ &= (\text{id} \otimes \Delta^0)(v_2)(x_0 \otimes 1^0 \otimes 1^0)((\rho \otimes \text{id}) \circ \rho)(x_0^*x_0)(x_0^* \otimes 1^0 \otimes 1^0) \\ &= (\text{id} \otimes \Delta^0)(v_2). \end{aligned}$$

Thus $(v_2 \otimes 1^0)(\rho_1 \otimes \text{id})(v_2) = (\text{id}_A \otimes \Delta^0)(v_2)$. Let $y = (\text{id}_A \otimes e)(v_2)$. Then

$$\begin{aligned} \rho_1(y) &= (\text{id}_A \otimes \text{id}_{H^0} \otimes e)((\rho_1 \otimes \text{id}_{H^0})(v_2)) \\ &= (\text{id}_A \otimes \text{id}_{H^0} \otimes e)((v_2^* \otimes 1^0)(\text{id}_A \otimes \Delta^0)(v_2)) \\ &= v_2^*[(\text{id}_A \otimes e)(v_2) \otimes 1^0] = v_2^*(y \otimes 1^0). \end{aligned}$$

Since $\|1 - y\| = \|(\text{id}_A \otimes e)(1 - v_2)\| \leq \|1 - v_2\| < 1$, y is invertible. Let $y = x|y|$ be the polar decomposition of y . Then x is a unitary element in A and

$$\rho_1(y) = v_2^*(y \otimes 1^0) = v_2^*(x \otimes 1^0)(|y| \otimes 1^0).$$

Hence

$$\rho_1(x)\rho_1(|y|) = v_2^*(x \otimes 1^0)(|y| \otimes 1^0).$$

Also,

$$\rho_1(y^*y) = (y^* \otimes 1^0)v_2v_2^*(y \otimes 1) = y^*y \otimes 1.$$

Thus $\rho_1(|y|) = |y| \otimes 1^0$. Hence $\rho_1(x) = v_2^*(x \otimes 1^0)$. It follows that

$$\begin{aligned} \text{Ad}(x \otimes 1^0) \circ \rho_1 \circ \text{Ad}(x^*) &= \text{Ad}((x \otimes 1^0)\rho_1(x^*)) \circ \rho_1 \\ &= \text{Ad}(v_2) \circ \rho_1 \\ &= \text{Ad}(v) \circ \rho = \sigma. \end{aligned}$$

Since $\rho_1 = \text{Ad}(x_0 \otimes 1^0) \circ \rho \circ \text{Ad}(x_0^*)$, we obtain the conclusion. \square

9. 2-COHOMOLOGY VANISHING THEOREM

Let (ρ, u) be a twisted coaction of H^0 on A with the Rohlin property. Let w be a unitary element in $(A^\infty \rtimes_{\rho^\infty, u} H) \otimes H$ satisfying Equation (5, 1)-(5, 3) and let \hat{w} be the unitary element in $\text{Hom}(H^0, A^\infty \rtimes_{\rho^\infty, u} H)$ induced by w . In this section, we shall show that there is a unitary element $x \in A \otimes H^0$ such that

$$(x \otimes 1^0)(\rho \otimes \text{id})(x)u(\text{id} \otimes \Delta^0)(x^*) = 1 \otimes 1^0 \otimes 1^0.$$

We recall that in Section 3, we construct a system of matrix units of $M_N(\mathbf{C})$,

$$\{(W_I^* \rtimes_{\hat{\rho}} 1^0)(1 \rtimes_{\rho, u} 1 \rtimes_{\hat{\rho}} \tau)(W_J \rtimes_{\hat{\rho}} 1^0)\}_{I, J \in \Lambda}$$

which is contained in $A^\infty \rtimes_{\rho^\infty, u} H$, where $W_I = \sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k$ for any $I = (i, j, k) \in \Lambda$. By Lemmas 5.3 and 6.1, we obtain the following lemma.

Lemma 9.1. *With the above notations and assumptions, $\{W_I^* \widehat{w}(\tau) W_J\}_{I,J \in \Lambda}$ is a system of matrix units of $M_N(\mathbf{C})$, which is contained in $A^\infty \rtimes_{\rho^\infty, u} H$.*

Proof. By the proof of Lemma 3.1, for any $I = (i, j, k), J = (s, t, r) \in \Lambda$,

$$W_I W_J^* = \sum_{t_2, t_3, j_3} \sqrt{d_k, d_r} \widehat{u}(w_{j_3 i}^k S(w_{t_2 t_3}^r), w_{st_2}^r)^* \rtimes_{\rho, u} w_{j_3 j}^k w_{t_3 t}^{r*}.$$

Hence by Lemma 5.3 and [10, Theorem 2.2],

$$\widehat{w}(\tau) W_I W_J^* \widehat{w}(\tau) = \sum_{t_2, t_3, j_3} \sqrt{d_k d_r} \tau(w_{j_3 j}^k w_{t_3 t}^{r*}) \widehat{u}^*(w_{i j_3}^k w_{t_2 t_3}^{r*}, w_{t_2 s}^r) \widehat{w}(\tau).$$

If $k \neq r$ or $j \neq t$, then $\widehat{w}(\tau) W_I W_J^* \widehat{w}(\tau) = 0$. We suppose that $k = r$ and $j = t$.

$$\widehat{w}(\tau) W_I W_J^* \widehat{w}(\tau) = \sum_{t_2, t_3} \widehat{u}^*(w_{it_3}^k S(w_{t_3 t_2}^k), w_{t_2 s}^k) \widehat{w}(\tau) = \epsilon(w_{is}^k) \widehat{w}(\tau) = \delta_{is} \widehat{w}(\tau),$$

where δ_{is} is the Kronecker delta. Thus for any $K, L, I, J \in \Lambda$,

$$W_K^* \widehat{w}(\tau) W_I W_J^* \widehat{w}(\tau) W_L = 0$$

if $I \neq J$. We suppose that $I = J$. Then since $\widehat{w}(\tau) W_I W_I^* \widehat{w}(\tau) = \widehat{w}(\tau)$,

$$W_K^* \widehat{w}(\tau) W_I W_I^* \widehat{w}(\tau) W_L = W_K^* \widehat{w}(\tau) W_L.$$

Furthermore,

$$\sum_{I \in \Lambda} W_I^* \widehat{w}(\tau) W_I W_I^* \widehat{w}(\tau) W_I = \sum_{I \in \Lambda} W_I^* \widehat{w}(\tau) W_I = 1$$

by Lemma 6.1. Therefore we obtain the conclusion. \square

We suppose that the C^* -Hopf algebra H^0 acts on a unital C^* -algebra \mathbf{C} trivially. Then by the discussions before Lemma 9.1, the set $\{(W_{0I}^* \rtimes_\Delta 1^0)(1 \rtimes 1 \rtimes_\Delta \tau)(W_{0J} \rtimes_\Delta 1^0)\}_{I, J \in \Lambda}$ is a system of matrix units of $\mathbf{C} \rtimes H \rtimes_\Delta H^0$ which is isomorphic to $M_N(\mathbf{C})$, where $W_{0I} = \sqrt{d_k} \rtimes w_{ij}^k \in \mathbf{C} \rtimes H$ for any $I = (i, j, k) \in \Lambda$. Thus we obtain the following homomorphism θ of $\mathbf{C} \rtimes H \rtimes_\Delta H^0$ into $A^\infty \rtimes_{\rho^\infty, u} H$. For any $I, J \in \Lambda$,

$$\theta((W_{0I}^* \rtimes_\Delta 1^0)(1 \rtimes 1 \rtimes_\Delta \tau)(W_{0J} \rtimes_\Delta 1^0)) = W_I^* \widehat{w}(\tau) W_J.$$

Lemma 9.2. *With the above notations, for any $h \in H$,*

$$\theta(1 \rtimes h) = \sum_{i, j, k} d_k (1 \rtimes_{\rho, u} w_{ij}^k)^* \widehat{w}(\tau) (1 \rtimes_{\rho, u} w_{ij}^k h).$$

Proof. Let h be any element in H . Then by Lemma 6.1,

$$\begin{aligned} 1 \rtimes h &= \sum_{I \in \Lambda} (W_I \rtimes_\Delta 1^0)^* (1 \rtimes 1 \rtimes_\Delta \tau) (W_I \rtimes_\Delta 1^0) (1 \rtimes h \rtimes_\Delta 1^0) \\ &= \sum_{i, j, k} d_k (1 \rtimes w_{ij}^k \rtimes_\Delta 1^0)^* (1 \rtimes 1 \rtimes_\Delta \tau) (1 \rtimes w_{ij}^k h \rtimes_\Delta 1^0). \end{aligned}$$

Since $\{w_{ij}^k\}$ is a system of comatrix units of H , for any i, j, k there are elements $(c_{ij}^k)_{st}^r \in \mathbf{C}$ such that $w_{ij}^k h = \sum_{st}^r (c_{ij}^k)_{st}^r w_{st}^r$. Hence

$$1 \rtimes h = \sum_{i, j, k, s, t, r} d_k (c_{ij}^k)_{st}^r (1 \rtimes w_{ij}^k \rtimes_\Delta 1^0)^* (1 \rtimes 1 \rtimes_\Delta \tau) (1 \rtimes w_{st}^r \rtimes_\Delta 1^0).$$

Thus by the definition of θ ,

$$\begin{aligned}
\theta(1 \rtimes h) &= \sum_{i,j,k,r,s,t} d_k(c_{ij}^k)_{st}^r (1 \rtimes_{\rho,u} w_{ij}^k)^* \widehat{w}(\tau) (1 \rtimes_{\rho,u} w_{st}^r) \\
&= \sum_{i,j,k} d_k(1 \rtimes_{\rho,u} w_{ij}^k)^* \widehat{w}(\tau) (1 \rtimes_{\rho,u} \sum_{s,t,r} (c_{ij}^k)_{st}^r w_{st}^r) \\
&= \sum_{i,j,k} d_k(1 \rtimes_{\rho,u} w_{ij}^k)^* \widehat{w}(\tau) (1 \rtimes_{\rho,u} w_{ij}^k h).
\end{aligned}$$

□

The restriction of θ to $1 \rtimes H$, the C^* -subalgebra of $\mathbf{C} \rtimes H \rtimes_{\Delta} H^0$ is a homomorphism of H to $A^\infty \rtimes_{\rho^\infty, u} H$. Hence there is a unitary element $v \in (A^\infty \rtimes_{\rho^\infty, u} H) \otimes H^0$ such that $\theta|_{1 \rtimes H} = \widehat{v}$. We recall the definitions V and \widehat{V} . Let \widehat{V} be a linear map from H to $A \rtimes_{\rho, u} H$ defined by $\widehat{V}(h) = 1 \rtimes_{\rho, u} h$ for any $h \in H$ and let V be the element in $(A \rtimes_{\rho, u} H) \otimes H^0$ induced by \widehat{V} . Then V and \widehat{V} are unitary elements in $(A \rtimes_{\rho, u} H) \otimes H^0$ and $\text{Hom}(H, A \rtimes_{\rho, u} H)$, respectively. Let x be a unitary element in $(A^\infty \rtimes_{\rho^\infty, u} H) \otimes H^0$ defined by $x = vV^*$.

Lemma 9.3. *With the above notations, $\widehat{x}(h) \in A^\infty$ for any $h \in H$.*

Proof. By Lemma 9.2 and [10, Theorem 2.2], for any $h \in H$,

$$\begin{aligned}
\widehat{x}(h) &= \widehat{v}(h_{(1)}) \widehat{V}(S(h_{(2)}^*))^* \\
&= \sum_{i,j,k} d_k(1 \rtimes_{\rho, u} w_{ij}^k)^* \widehat{w}(\tau) (1 \rtimes_{\rho, u} w_{ij}^k h_{(1)}) (1 \rtimes_{\rho, u} S(h_{(2)}^*))^* \\
&= \sum_{i,j,k,j_1,j_2} d_k(\widehat{u}(S(w_{j_1 j_2}^k), w_{ij_1}^k)^* \rtimes_{\rho, u} w_{j_2 j}^{k*}) (\widehat{w}(\tau) \rtimes_{\rho, u} w_{ij}^k h_{(1)}) \\
&\quad \times (\widehat{u}(h_{(3)}^*), S(h_{(4)}^*))^* \rtimes_{\rho, u} S(h_{(2)})) \\
&= \sum_{i,j,k,j_1,j_2,j_3,j_4,i_1} d_k(\widehat{u}(S(w_{j_1 j_2}^k), w_{ij_1}^k)^* [w_{j_2 j_3}^{k*} \cdot_{\rho, u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i i_1}^k h_{(1)})) \\
&\quad \rtimes_{\rho, u} w_{j_4 j}^{k*} w_{i_1 j}^k h_{(2)}) (\widehat{u}(h_{(4)}^*), S(h_{(5)}^*))^* \rtimes_{\rho, u} S(h_{(3)})) \\
&= \sum_{i,j,k,j_1,j_2,j_4,i_1} d_k(\widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k) [w_{j_2 j_3}^{k*} \cdot_{\rho, u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i i_1}^k h_{(1)})) \\
&\quad \rtimes_{\rho, u} S(w_{j_4 j}^k) w_{i_1 j}^k h_{(2)}) (\widehat{u}^*(S(h_{(4)}), h_{(5)}) \rtimes_{\rho, u} S(h_{(3)})) \\
&= \sum_{i,k,j_1,j_2,j_3,j_4} d_k \widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k) [w_{j_2 j_3}^{k*} \cdot_{\rho, u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i j_4}^k h_{(1)}) \rtimes_{\rho, u} h_{(2)}) \\
&\quad \times (\widehat{u}^*(S(h_{(4)}), h_{(5)}) \rtimes_{\rho, u} S(h_{(3)})) \\
&= \sum_{i,k,j_1,j_2,j_3,j_4} d_k \widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k) [w_{j_2 j_3}^{k*} \cdot_{\rho, u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i j_4}^k h_{(1)}) \\
&\quad \times [h_{(2)} \cdot_{\rho, u} \widehat{u}^*(S(h_{(7)}), h_{(8)})] \widehat{u}(h_{(3)}, S(h_{(6)})) \rtimes_{\rho, u} h_{(4)} S(h_{(5)}) \\
&= \sum_{i,k,j_1,j_2,j_3,j_4} d_k \widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k) [w_{j_2 j_3}^{k*} \cdot_{\rho, u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i j_4}^k h_{(1)}) \\
&\quad \times [h_{(2)} \cdot_{\rho, u} \widehat{u}^*(S(h_{(5)}), h_{(6)})] \widehat{u}(h_{(3)}, S(h_{(4)})).
\end{aligned}$$

Furthermore, using the Equations (1) and (2) in Section 2, we can see that for any $h \in H$,

$$\widehat{x}(h) = \sum_{i,k,j_1,j_2,j_3,j_4} d_k \widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k) [w_{j_2 j_3}^{k*} \cdot_{\rho, u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i j_4}^k h).$$

Since $w_{j_2 j_3}^{k*} \cdot_{\rho, u} \widehat{w}(\tau) \in A^\infty$ for any j_2, j_3, k , we obtain the conclusion. □

By the above lemma, we can see that x is a unitary element in $A^\infty \otimes H^0$. We recall that $\rho_{H^0}^{A \rtimes_{\rho,u} H}$ is the trivial coaction of H^0 on $A \rtimes_{\rho,u} H$ defined by $\rho_{H^0}^{A \rtimes_{\rho,u} H}(a) = a \otimes 1^0$ for any $a \in A \rtimes_{\rho,u} H$. Also, we note that $\rho = \text{Ad}(V) \circ \rho_{H^0}^{A \rtimes_{\rho,u} H}$ by [6, Lemma 3.12], where we regard A as a C^* -subalgebra of $A \rtimes_{\rho,u} H$. Furthermore, since \widehat{v} is a homomorphism of H to $A^\infty \rtimes_{\rho^\infty,u} H$,

$$(v \otimes 1^0)(\rho_{H^0}^{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \text{id})(v) = (\text{id} \otimes \Delta^0)(v).$$

Proposition 9.4. *With the above notations,*

$$(x \otimes 1^0)(\rho^\infty \otimes \text{id})(x)u(\text{id} \otimes \Delta^0)(x^*) = 1 \otimes 1^0 \otimes 1^0.$$

Proof. Since $x = vV^*$ and $\rho = \text{Ad}(V) \circ \rho_{H^0}^{A \rtimes_{\rho,u} H}$,

$$\begin{aligned} & (\rho^\infty \otimes \text{id})(x^*)(x^* \otimes 1^0)(\text{id} \otimes \Delta^0)(x) \\ &= (\rho^\infty \otimes \text{id})(Vv^*)(Vv^* \otimes 1^0)(\text{id} \otimes \Delta^0)(vV^*) \\ &= (V \otimes 1^0)(\rho_{H^0}^{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \text{id})(Vv^*)(v^* \otimes 1^0)(\text{id} \otimes \Delta^0)(vV^*). \end{aligned}$$

Since $(v \otimes 1^0)(\rho_{H^0}^{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \text{id})(v) = (\text{id} \otimes \Delta^0)(v)$,

$(\rho^\infty \otimes \text{id})(x^*)(x^* \otimes 1^0)(\text{id} \otimes \Delta^0)(x) = (V \otimes 1^0)(\rho_{H^0}^{A \rtimes_{\rho,u} H} \otimes \text{id})(V)(\text{id} \otimes \Delta^0)(V^*) = u$ by [6, Lemma 3.12]. \square

We recall that $\{\phi_{ij}^k\}$ is a system of matrix units of H^0 .

Lemma 9.5. *Let (ρ, u) be a twisted coaction of H^0 on A with the Rohlin property. Then for any $\epsilon > 0$, there is a unitary element $x \in A \otimes H^0$ satisfying that*

$$\begin{aligned} & \| (x \otimes 1^0)(\rho \otimes \text{id})(x)u(\text{id} \otimes \Delta^0)(x^*) - 1 \otimes 1^0 \otimes 1^0 \| < \epsilon \\ & \| x - 1 \otimes 1^0 \| < \epsilon + L \| u - 1 \otimes 1^0 \otimes 1^0 \|, \end{aligned}$$

where

$$L = \max\left\{ \sum_{i,j,k,t,r,t_1,t_2,j_1} d_k \| \widehat{V}(w_{ij}^k)^* \| \| \widehat{V}(w_{j_1 j}^k w_{t_2 t_1}^r) \widehat{V}^*(w_{t_1 t}^r) \|, 1 \right\}$$

Proof. Modifying the proof of Izumi [4, Lemma 3.12], we shall prove this lemma. By Proposition 9.4, there is a unitary element $x_0 \in A^\infty \otimes H^0$ satisfying that

$$(x_0 \otimes 1^0)(\rho^\infty \otimes \text{id})(x_0)u(\text{id} \otimes \Delta^0)(x_0^*) = 1 \otimes 1^0 \otimes 1^0.$$

By the proof of Lemma 9.3, for any $h \in H$,

$$\widehat{x_0}(h) = \sum_{i,j,k} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) \widehat{V}(w_{ij}^k h_{(1)}) \widehat{V}^*(h_{(2)}).$$

Thus

$$x_0 = \sum_{i,j,k,s,t,r,t_1} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) \widehat{V}(w_{ij}^k w_{st_1}^r) \widehat{V}^*(w_{t_1 t}^r) \otimes \phi_{st}^r.$$

Since $\sum_{i,j,k} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) \widehat{V}(w_{ij}^k) = 1$ by Lemma 6.1,

$$1 \otimes 1^0 = \sum_{i,j,k,s,t,r,t_1} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) \widehat{V}(w_{ij}^k) \widehat{V}(w_{st_1}^r) \widehat{V}^*(w_{t_1 t}^r) \otimes \phi_{st}^r.$$

Thus

$$x_0 - 1 \otimes 1^0 = \sum_{i,j,k,s,t,r,t_1} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) [\widehat{V}(w_{ij}^k w_{st_1}^r) - \widehat{V}(w_{ij}^k) \widehat{V}(w_{st_1}^r)] \widehat{V}^*(w_{t_1 t}^r) \otimes \phi_{st}^r.$$

Since $u = (V \otimes 1^0)(\rho_{H^0}^{A \rtimes_{\rho,u} H} \otimes \text{id})(V)(\text{id} \otimes \Delta^0)(V^*)$ by [6, Lemma 3.12],

$$\widehat{V}(w_{ij}^k) \widehat{V}(w_{st_1}^r) = \sum_{j_1, t_2} \widehat{u}(w_{ij_1}^k, w_{st_2}^r) \widehat{V}(w_{j_1 j}^k w_{t_2 t_1}^r)$$

for any i, j, k, s, t_1, r . Hence

$$\begin{aligned} x_0 - 1 \otimes 1^0 &= \sum_{i,j,k,s,t,r,t_1,t_2,j_1} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) [\epsilon(w_{ij_1}^k) \epsilon(w_{st_2}^r) - \widehat{u}(w_{ij_1}^k, w_{st_2}^r)] \\ &\quad \times \widehat{V}(w_{j_1 j}^k w_{t_2 t_1}^r) \widehat{V}^*(w_{t_1 t}^r) \otimes \phi_{st}^r. \end{aligned}$$

Since $\|\epsilon(w_{ij_1}^k) \epsilon(w_{st_2}^r) - \widehat{u}(w_{ij_1}^k, w_{st_2}^r)\| \leq \|1 \otimes 1^0 \otimes 1^0 - u\|$ for any i, j_1, k, s, t_2, r ,

$$\begin{aligned} \|x_0 - 1 \otimes 1 \otimes 1^0\| &\leq \sum_{i,j,k,s,t,r,t_1,t_2,j_1} d_k \|\widehat{V}(w_{ij}^k)\| \|\widehat{V}(w_{j_1 j}^k w_{t_2 t_1}^r) \widehat{V}^*(w_{t_1 t}^r)\| \\ &\quad \times \|1 \otimes 1^0 \otimes 1^0 - u\|. \end{aligned}$$

Since x_0 is a unitary element in $A^\infty \otimes H^0$, we can choose a desired unitary element x in $A \otimes H^0$. \square

Theorem 9.6. *Let (ρ, u) be a twisted coaction of a finite dimensional C^* -Hopf algebra H^0 on a unital C^* -algebra A with the Rohlin property. Then there is a unitary element $x \in A \otimes H^0$ such that*

$$(x \otimes 1^0)(\rho \otimes \text{id})(x)u(\text{id} \otimes \Delta^0)(x^*) = 1 \otimes 1^0 \otimes 1^0.$$

Proof. We shall prove this lemma modifying the proof of [4, Lemma 3.12]. Let $u_0 = u$ and $\rho_0 = \rho$. By Lemma 9.5, for $\frac{1}{2L}$, there is a unitary element $y_0 \in A \otimes H^0$ such that

$$\|1 \otimes 1^0 \otimes 1^0 - (y_0 \otimes 1^0)(\rho_0 \otimes \text{id})(y_0)u_0(\text{id} \otimes \Delta^0)(y_0^*)\| < \frac{1}{2L} < \frac{1}{2}.$$

Let

$$\rho_1 = \text{Ad}(y_0) \circ \rho_0, \quad u_1 = (y_0 \otimes 1^0)(\rho_0 \otimes \text{id})(y_0)u_0(\text{id} \otimes \Delta^0)(y_0^*).$$

Then since (ρ_1, u_1) is a twisted coaction of H^0 on A which is exterior equivalent to (ρ_0, u_0) , by Proposition 5.4 (ρ_1, u_1) has the Rohlin property. Thus by Lemma 9.5, for $\frac{1}{(2L)^2}$, there is a unitary element $y_1 \in A \otimes H^0$ such that

$$\begin{aligned} \|1 \otimes 1^0 \otimes 1^0 - (y_1 \otimes 1^0)(\rho_1 \otimes \text{id})(y_1)u_1(\text{id} \otimes \Delta^0)(y_1^*)\| &< \frac{1}{(2L)^2} < \frac{1}{2^2}, \\ \|y_1 - 1 \otimes 1^0\| &< \frac{1}{(2L)^2} + L\|u_1 - 1 \otimes 1^0 \otimes 1^0\| < \frac{1}{(2L)^2} + \frac{1}{2} < \frac{1}{2^2} + \frac{1}{2} = \frac{3}{2^2} \end{aligned}$$

since $u_1 = (y_0 \otimes 1^0)(\rho_0 \otimes \text{id})(y_0)u_0(\text{id} \otimes \Delta^0)(y_0^*)$. Let

$$\rho_2 = \text{Ad}(y_1) \circ \rho_1, \quad u_2 = (y_1 \otimes 1^0)(\rho_1 \otimes \text{id})(y_1)u_1(\text{id} \otimes \Delta^0)(y_1^*).$$

Then since (ρ_2, u_2) is a twisted coaction of H^0 on A which is exterior equivalent to (ρ_1, u_1) , by Proposition 5.4 (ρ_2, u_2) has the Rohlin property. Thus by Lemma 9.5, for $\frac{1}{(2L)^3}$, there is a unitary element $y_2 \in A \otimes H^0$ such that

$$\begin{aligned} \|1 \otimes 1^0 \otimes 1^0 - (y_2 \otimes 1^0)(\rho_2 \otimes \text{id})(y_2)u_2(\text{id} \otimes \Delta^0)(y_2^*)\| &< \frac{1}{(2L)^3} < \frac{1}{2^3}, \\ \|y_2 - 1 \otimes 1^0\| &< \frac{1}{(2L)^3} + L\|u_2 - 1 \otimes 1^0 \otimes 1^0\| < \frac{1}{(2L)^3} + \frac{1}{2^2} < \frac{1}{2^3} + \frac{1}{2^2} = \frac{3}{2^3}. \end{aligned}$$

It follows that by induction that there are sequences $\{(\rho_n, u_n)\}$ of twisted coactions of H^0 on A and $\{y_n\}$ of unitary elements in $A \otimes H^0$ satisfying that for any $n \in \mathbb{N}$,

$$\|1 \otimes 1^0 \otimes 1^0 - u_n\| < \frac{1}{(2L)^n} < \frac{1}{2^n}, \quad \|1 \otimes 1^0 - y_n\| < \frac{1}{2^{n+1}} + \frac{1}{2^n} = \frac{3}{2^{n+1}}.$$

Let $x_n = y_n y_{n-1} \cdots y_0 \in A \otimes H^0$ for any $n \in \mathbb{N} \cup \{0\}$. Then x_n is a unitary element in $A \otimes H^0$ satisfying that

$$u_{n+1} = (x_n \otimes 1^0)(\rho \otimes \text{id})(x_n)u_0(\text{id} \otimes \Delta^0)(x_n^*)$$

for any $\in \mathbb{N} \cup \{0\}$ by routine computations. Furthermore,

$$\|u_n - 1 \otimes 1^0 \otimes 1^0\| < \frac{1}{2^n} \longrightarrow 0 \quad (n \longrightarrow +\infty)$$

Also, since by easy computations, we see that $\{x_n\}$ is a Cauchy sequence, there is a unitary element $x \in A \otimes H^0$ such that $x_n \longrightarrow x$ ($n \longrightarrow +\infty$). Therefore, we obtain that

$$1 \otimes 1^0 \otimes 1^0 = (x \otimes 1^0)(\rho \otimes \text{id})(x)u(\text{id} \otimes \Delta^0)(x^*).$$

□

10. APPROXIMATELY UNITARY EQUIVALENCE OF COACTIONS

Let ρ be a coaction of H^0 on A with the Rohlin property. Let w be a unitary element in $(A \rtimes_{\rho} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for $\hat{\rho}$. Let (ρ_1, u) be a twisted coaction H^0 on A which is exterior equivalent to ρ . Let v be a unitary element in $A \otimes H^0$ satisfying Conditions (1), (2) in Definition 2.2, that is,

- (1) $\rho_1 = \text{Ad}(v) \circ \rho$,
- (2) $u = (v \otimes 1)(\rho \otimes \text{id})(v)(\text{id} \otimes \Delta)(v^*)$.

By Proposition 5.4, (ρ_1, u) has the Rohlin property. Let w_1 be a unitary element in $(A \rtimes_{\rho_1, u} H) \otimes H$ satisfying Equations (5, 1)-(5, 3) for $\hat{\rho}_1$. By Lemma 5.5, $\hat{w}(\tau) = \hat{w}_1(\tau)$. Let

$$x = N(\text{id} \otimes e)(v\rho^\infty(\hat{w}(\tau))) = N\hat{v}(e_{(1)})[e_{(2)} \cdot_{\rho^\infty} \hat{w}(\tau)].$$

We have the following lemma which is similar to Lemma 8.1.

Lemma 10.1. *With the above notations and assumptions, x is a unitary element in A^∞ .*

Proof. In the same way as in the proof of Lemma 8.1, we can see that $xx^* = 1$. Next we shall show that $x^*x = 1$. Let $f = e$.

$$\begin{aligned} x^*x &= N^2[e_{(2)} \cdot_{\rho^\infty} \hat{w}(\tau)]^* \hat{v}(e_{(1)})^* \hat{v}(f_{(1)})[f_{(2)} \cdot_{\rho^\infty} \hat{w}(\tau)] \\ &= N^2[S(e_{(2)}^*) \cdot_{\rho^\infty} \hat{w}(\tau)] \hat{v}^*(S(e_{(1)}^*)) \hat{v}(f_{(1)})[f_{(2)} \cdot_{\rho^\infty} \hat{w}(\tau)] \\ &= N^2 \hat{V}(S(e_{(3)}^*)) \hat{w}(\tau) \hat{V}^*(S(e_{(2)}^*)) \hat{v}^*(S(e_{(1)}^*)) \hat{v}(f_{(1)}) \hat{V}(f_{(2)}) \hat{w}(\tau) \hat{V}^*(f_{(3)}) \\ &= N^2 \hat{V}(S(e_{(3)}^*)) \hat{w}(\tau) (1 \rtimes_{\rho} e_{(2)}^*) \hat{v}^*(S(e_{(1)}^*)) \hat{v}(f_{(1)}) (1 \rtimes_{\rho} f_{(2)}) \hat{w}(\tau) \hat{V}^*(f_{(3)}) \\ &= N^2 \hat{V}(S(e_{(4)}^*)) \hat{w}(\tau) ([e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(f_{(1)})] \rtimes_{\rho} e_{(3)}^* f_{(2)}) \hat{w}(\tau) \hat{V}^*(f_{(3)}) \\ &= N^2 \hat{V}(S(e_{(4)}^*)) [e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(f_{(1)})] \tau(e_{(3)}^* f_{(2)}) \hat{w}(\tau) \hat{V}^*(f_{(3)}) \\ &= N^2 \hat{V}(S(e_{(6)}^*)) [e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(S(e_{(3)}^*) e_{(4)}^* f_{(1)})] \tau(e_{(5)}^* f_{(2)}) \hat{w}(\tau) \hat{V}^*(f_{(3)}) \\ &= N^2 \hat{V}(S(e_{(5)}^*)) [e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(S(e_{(3)}^*))] \tau(e_{(4)}^* f_{(1)}) \hat{w}(\tau) \hat{V}^*(f_{(2)}) \\ &= N^2 \hat{V}(S(e_{(7)}^*)) [e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(S(e_{(3)}^*))] \tau(e_{(4)}^* f_{(1)}) \hat{w}(\tau) \hat{V}^*(S(e_{(6)}^*) e_{(5)}^* f_{(2)}) \\ &= N^2 \hat{V}(S(e_{(6)}^*)) [e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(S(e_{(3)}^*))] \tau(e_{(4)}^* f_{(1)}) \hat{w}(\tau) \hat{V}^*(S(e_{(5)}^*)) \\ &= N \hat{V}(S(e_{(5)}^*)) [e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(S(e_{(3)}^*))] \hat{w}(\tau) \hat{V}^*(S(e_{(4)}^*)) \\ &= N[S(e_{(4)}^*) \cdot_{\rho^\infty} [e_{(2)}^* \cdot_{\rho} \hat{v}^*(S(e_{(1)}^*)) \hat{v}(S(e_{(3)}^*))] \hat{w}(\tau)]. \end{aligned}$$

Let E^{ρ^∞} be the conditional expectation from A^∞ onto $(A^\rho)^\infty$. Then since $e = \sum_{i,k} \frac{d_k}{N} w_{ii}^k$,

$$\begin{aligned}
E^{\rho^\infty}(x^*x) &= f \cdot_{\rho^\infty} x^*x = N[fS(e_{(4)}^*) \cdot_{\rho^\infty} [e_{(2)}^* \cdot_{\rho} \widehat{v^*}(S(e_{(1)}^*))\widehat{v}(S(e_{(3)}^*))]\widehat{w}(\tau)] \\
&= N[f \cdot_{\rho^\infty} [e_{(2)}^* \cdot_{\rho} \widehat{v^*}(S(e_{(1)}^*))\widehat{v}(S(e_{(3)}^*))]\widehat{w}(\tau)] \\
&= \sum_{i,j,j_1,k} d_k [f \cdot_{\rho^\infty} [w_{jj_1}^{k*} \cdot_{\rho} \widehat{v^*}(S(w_{ij}^{k*}))\widehat{v}(S(w_{j_1i}^{k*}))]\widehat{w}(\tau)] \\
&= \sum_{i,j,j_1,k} d_k [f \cdot_{\rho^\infty} [w_{jj_1}^{k*} \cdot_{\rho} \widehat{v^*}(w_{ji}^k)\widehat{v}(w_{j_1i}^k)]\widehat{w}(\tau)] \\
&= \sum_{j,j_1,k} d_k [f \cdot_{\rho^\infty} [w_{jj_1}^{k*} \cdot_{\rho} \epsilon(w_{j_1j}^k)]\widehat{w}(\tau)] = \sum_{j,k} d_k [f \cdot_{\rho^\infty} [w_{jj}^{k*} \cdot_{\rho} 1]\widehat{w}(\tau)] \\
&= N[f \cdot_{\rho^\infty} \epsilon(e)\widehat{w}(\tau)] = N[f \cdot_{\rho^\infty} \widehat{w}(\tau)] = 1
\end{aligned}$$

by Lemma 5.6. Since E^{ρ^∞} is faithful, we obtain the conclusion. \square

Definition 10.1. Coactions ρ and σ of H^0 on A are *approximately unitarily equivalent* if there is a unitary element $v \in A^\infty \otimes H^0$ such that

$$\sigma(a) = v\rho(a)v^*$$

for any $a \in A$.

Let ρ and σ be coactions of H^0 on A which are approximately unitarily equivalent. Then there is a unitary element v in $A^\infty \otimes H^0$ such that $\sigma(a) = v\rho(a)v^*$ for any $a \in A$. We write $v = (v_n)$, where v_n is a unitary element in A . Then since $a(\text{id} \otimes \epsilon^0)(v) = (\text{id} \otimes \epsilon^0)(v)a$ for any $a \in A$, $(\text{id} \otimes \epsilon^0)(v)$ is a unitary element in A_∞ . Let $z = (\text{id} \otimes \epsilon^0)(v)$ and $w = v(z^* \otimes 1^0)$. Then w is a unitary element in $A^\infty \otimes H^0$ and

$$w\rho(a)w^* = v(z^* \otimes 1^0)\rho(a)(z \otimes 1^0)v^* = v\rho(a)v^* = \sigma(a)$$

for any $a \in A$. Furthermore, $(\text{id} \otimes \epsilon^0)(w) = zz^* = 1$. Hence if we write $w = (w_n)$, where w_n is a unitary element in $A \otimes H^0$, then $w_n = v_n((\text{id} \otimes \epsilon^0)(v_n^*) \otimes 1^0)$. Thus $(\text{id} \otimes \epsilon)(w_n) = 1$. Therefore, we may assume that $(\text{id} \otimes \epsilon^0)(v_n) = 1$ for any $n \in \mathbb{N}$. We shall show the following lemma.

Lemma 10.2. *Let σ and ρ be coactions of H^0 on A . We suppose that ρ has the Rohlin property and that σ is approximately unitary equivalent to ρ . Then for each finite subset F of A and any positive number $\epsilon > 0$, there is a unitary element $x \in A$ such that*

$$\begin{aligned}
&\|\sigma(a) - (\text{Ad}(x \otimes 1^0) \circ \rho \circ \text{Ad}(x^*))(a)\| < \epsilon, \\
&\|xa - ax\| < \epsilon + L \max_{i,j,k} \|\sigma([S(w_{ij}^k) \cdot_{\rho} a]) - \rho([S(w_{ij}^k) \cdot_{\rho} a])\|
\end{aligned}$$

for any $a \in F$, where $L = \sum_{i,j,k} d_k \|\text{id} \otimes w_{ij}^k\|$.

We shall prove this lemma by showing a series of several lemmas. Since ρ and σ are approximately unitarily equivalent, there is a unitary element $v_0 \in A^\infty \otimes H^0$ such that $\sigma(a) = v_0\rho(a)v_0^*$ for any $a \in A$. Let F be any finite subset of A and ϵ any positive number. Then there is a unitary element $v \in A \otimes H^0$ with $(\text{id} \otimes \epsilon^0)(v) = 1$ such that

$$\begin{aligned}
&\|\sigma(a) - v\rho(a)v^*\| < \epsilon, \\
&\|\sigma([S(w_{ij}^k) \cdot_{\sigma} a]) - v\rho([S(w_{ij}^k) \cdot_{\sigma} a])v^*\| < \epsilon, \\
&\|\sigma([S(w_{ij}^k) \cdot_{\rho} a]) - v\rho([S(w_{ij}^k) \cdot_{\rho} a])v^*\| < \epsilon
\end{aligned}$$

for any $a \in F$ and $i, j = 1, 2, \dots, d_k, k = 1, 2, \dots, K$. Let $x = N(\text{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau)))$. Let $\rho_1 = \text{Ad}(v) \circ \rho$ and $u = (v \otimes 1^0)(\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(v^*)$. Then (ρ_1, u) is a twisted coaction of H^0 on A which is exterior equivalent to ρ . Hence by Lemma 10.1, x is a unitary element in A^∞ .

Lemma 10.3. *With the above notations and assumptions, for any $a \in F$,*

$$\|\rho(x)(x^* \otimes 1^0)v\rho(a) - N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v\| < N\epsilon.$$

Proof. We note that

$$\begin{aligned} x &= N(\text{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))) = \sum_{i,k} d_k(\text{id} \otimes w_{ii}^k)(v\rho^\infty(\widehat{w}(\tau))) \\ &= \sum_{i,j,k} d_k \widehat{v}(w_{ij}^k)[w_{ji}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]. \end{aligned}$$

Also, $x^* = N[e_{(1)} \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(e_{(2)})$ since $x = N(\text{id} \otimes S(e^*))(\rho^\infty(\widehat{w}(\tau)))$. Then by Lemma 5.3 for any $h \in H$,

$$\begin{aligned} (\rho(x)(x^* \otimes 1^0)v\rho(a))\widehat{}(h) &= [h_{(1)} \cdot_{\rho^\infty} x]x^*\widehat{v}(h_{(2)})[h_{(3)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)][h_{(2)}w_{ji}^k \cdot_{\rho^\infty} \widehat{w}(\tau)][e_{(1)} \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(e_{(2)})\widehat{v}(h_{(3)})[h_{(4)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k,t} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(3)}w_{ti}^k)\widehat{V}(e_{(1)})\widehat{w}(\tau)\widehat{V}^*(e_{(2)})\widehat{v}^*(e_{(3)}) \\ &\quad \times \widehat{v}(h_{(4)})[h_{(5)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k,t} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\widehat{w}(\tau)(1 \rtimes_{\rho} S(h_{(3)}w_{ti}^k)e_{(1)})\widehat{w}(\tau)\widehat{V}^*(e_{(2)})\widehat{v}^*(e_{(3)}) \\ &\quad \times \widehat{v}(h_{(4)})[h_{(5)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k,t} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\tau(S(h_{(3)}w_{ti}^k)e_{(1)})\widehat{w}(\tau)\widehat{V}^*(e_{(2)})\widehat{v}^*(e_{(3)}) \\ &\quad \times \widehat{v}(h_{(4)})[h_{(5)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k,t,t_1,t_2} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(3)}w_{tt_1}^k S(h_{(4)}w_{t_1 t_2}^k)e_{(2)}) \\ &\quad \times \tau(S(h_{(5)}w_{t_2 i}^k)e_{(1)})\widehat{v}^*(e_{(3)})\widehat{v}(h_{(6)})[h_{(7)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k,t,t_1} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(3)}w_{tt_1}^k)\tau(S(h_{(4)}w_{t_1 i}^k)e_{(1)})\widehat{v}^*(e_{(2)}) \\ &\quad \times \widehat{v}(h_{(5)})[h_{(6)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k,t,t_1,t_2,t_3} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(3)}w_{tt_1}^k) \\ &\quad \times \widehat{v}^*(h_{(4)}w_{t_1 t_2}^k S(h_{(5)}w_{t_2 t_3}^k)e_{(2)})\tau(S(h_{(6)}w_{t_3 i}^k)e_{(1)})\widehat{v}(h_{(7)})[h_{(8)} \cdot_{\rho} a] \\ &= N \sum_{i,j,k,t,t_1,t_2} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(3)}w_{tt_1}^k)\widehat{v}^*(h_{(4)}w_{t_1 t_2}^k) \\ &\quad \times \tau(S(h_{(5)}w_{t_2 i}^k)e_{(1)})\widehat{v}(h_{(6)})[h_{(7)} \cdot_{\rho} a] \\ &= \sum_{i,j,k,t,t_1} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(3)}w_{tt_1}^k)\widehat{v}^*(h_{(4)}w_{t_1 i}^k)\widehat{v}(h_{(5)})[h_{(6)} \cdot_{\rho} a] \\ &= \sum_{i,j,k,t_1} d_k [h_{(1)} \cdot_{\rho} \widehat{v}(w_{ij}^k)][h_{(2)}w_{jt_1}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(3)}w_{t_1 i}^k)\widehat{v}(h_{(4)})[h_{(5)} \cdot_{\rho} a]. \end{aligned}$$

Thus

$$\begin{aligned}\rho(x)(x^* \otimes 1^0)v\rho(a) &= \sum_{i,k} d_k(\text{id} \otimes w_{ii}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))v\rho(a) \\ &= N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))v\rho(a).\end{aligned}$$

Hence

$$\begin{aligned}& \|\rho(x)(x^* \otimes 1^0)v\rho(a) - N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v\| \\ &= N\|(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))(\rho(a) - \sigma(a)v)\| \\ &\leq N\|\rho(a) - \sigma(a)v\| = N\|\rho(a)v^* - \sigma(a)\| < N\epsilon.\end{aligned}$$

□

Lemma 10.4. *With the above notations and assumptions, for any $a \in F$,*

$$\begin{aligned}& \|N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v \\ & - \sum_{i,j,k} d_k(\text{id} \otimes w_{ij}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))\rho([S(w_{ji}^k) \cdot_\sigma a]v^*))v)\| < L\epsilon,\end{aligned}$$

where $L = \sum_{i,j,k} d_k \|\text{id} \otimes w_{ij}^k\|$.

Proof. Since $e = \sum_{i,k} \frac{d_k}{N} w_{ii}^k$,

$$\begin{aligned}& N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v \\ &= \sum_{i,k} d_k(\text{id} \otimes w_{ii}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v.\end{aligned}$$

Thus for any $h \in H$

$$\begin{aligned}& [N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v]\widehat{v}(h) \\ &= \sum_{i,j,k,t_1} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(2)}w_{jt_1}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(3)}w_{t_1i}^k)[h_{(4)} \cdot_\sigma a]\widehat{v}(h_{(5)}) \\ &= \sum_{i,j,k,t_1,t_2} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(2)}w_{jt_1}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(3)}w_{t_1t_2}^k)[h_{(4)}\epsilon(w_{t_2i}^k) \cdot_\sigma a]\widehat{v}(h_{(5)}) \\ &= \sum_{i,j,k,t_1,t_2,t_3} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(2)}w_{jt_1}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(3)}w_{t_1t_2}^k) \\ &\quad \times [h_{(4)}w_{t_2t_3}^k \cdot_\sigma [S(w_{t_3i}^k) \cdot_\sigma a]]\widehat{v}(h_{(5)}).\end{aligned}$$

Thus

$$\begin{aligned}& N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v \\ &= \sum_{i,t_3,k} d_k(\text{id} \otimes w_{it_3}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*\sigma([S(w_{t_3i}^k) \cdot_\sigma a]))v).\end{aligned}$$

Hence

$$\begin{aligned}
& \|N(\text{id} \otimes e)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v \\
& - \sum_{i,t_3,k} d_k(\text{id} \otimes w_{it_3}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))\rho([S(w_{t_3i}^k) \cdot_\sigma a]v^*))v\| \\
& = \| \sum_{i,t_3,k} d_k(\text{id} \otimes w_{it_3}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*\sigma([S(w_{t_3i}^k) \cdot_\sigma a] \\
& - \rho([S(w_{t_3i}^k) \cdot_\sigma a]v^*))v\| \\
& \leq \sum_{i,t_3,k} d_k \|\text{id} \otimes w_{it_3}^k\| \|v^*\sigma([S(w_{t_3i}^k) \cdot_\sigma a]) - \rho([S(w_{t_3i}^k) \cdot_\sigma a]v^*\| \\
& < \sum_{i,t_3,k} d_k \|\text{id} \otimes w_{it_3}^k\| \epsilon < L\epsilon.
\end{aligned}$$

□

Lemma 10.5. *With the above notations and assumptions, for any $a \in A$,*

$$\begin{aligned}
& \sum_{i,j,k} d_k(\text{id} \otimes w_{ij}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))\rho([S(w_{t_3i}^k) \cdot_\sigma a]v^*))v \\
& = \rho(a)\rho^\infty(x)(x^* \otimes 1^0)v.
\end{aligned}$$

Proof. We shall show the above equation by routine computations. For any $h \in H$

$$\begin{aligned}
& [\sum_{i,t_3,k} d_k(\text{id} \otimes w_{it_3}^k)((\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau))\rho([S(w_{t_3i}^k) \cdot_\sigma a]v^*))v]\widehat{h} \\
& = \sum_{i,j,k,t_1,t_2,t_3} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(2)}w_{jt_1}^k \cdot_{\rho^\infty} \widehat{w}(\tau)][h_{(3)}w_{t_1t_2}^k \cdot_\rho [S(w_{t_3i}^k) \cdot_\sigma a]] \\
& \times \widehat{v}^*(h_{(4)}w_{t_2t_3}^k)\widehat{v}(h_{(5)}) \\
& = \sum_{i,j,k,t_2,t_3} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(2)}w_{jt_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)[S(w_{t_3i}^k) \cdot_\sigma a]]\widehat{v}^*(h_{(3)}w_{t_2t_3}^k)\widehat{v}(h_{(4)}) \\
& = \sum_{i,j,k,t_2,t_3} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(2)}w_{jt_2}^k \cdot_{\rho^\infty} [S(w_{t_3i}^k) \cdot_\sigma a]\widehat{w}(\tau)]\widehat{v}^*(h_{(3)}w_{t_2t_3}^k)\widehat{v}(h_{(4)}) \\
& = \sum_{i,j,k,t_2,t_3} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][w_{jt_2}^k \cdot_{\rho^\infty} [S(w_{t_3i}^k) \cdot_\sigma a]\widehat{w}(\tau)]\widehat{v}^*(h_{(2)}w_{t_2t_3}^k)\widehat{v}(h_{(3)}) \\
& = \sum_{i,j,k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho \widehat{v}(w_{ij}^k)][w_{jj_1}^k \cdot_\rho [S(w_{t_3i}^k) \cdot_\sigma a]][w_{j_1t_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(2)}w_{t_2t_3}^k) \\
& \times \widehat{v}(h_{(3)}) \\
& = \sum_{i,j,k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho [w_{ij}^k \cdot_\sigma [S(w_{t_3i}^k) \cdot_\sigma a]]\widehat{v}(w_{jj_1}^k)][w_{j_1t_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(2)}w_{t_2t_3}^k) \\
& \times \widehat{v}(h_{(3)}) \\
& = \sum_{j,k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho [\epsilon(w_{t_3j}^k) \cdot_\sigma a]\widehat{v}(w_{jj_1}^k)][w_{j_1t_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(2)}w_{t_2t_3}^k)\widehat{v}(h_{(3)}) \\
& = \sum_{k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho a\widehat{v}(w_{t_3j_1}^k)][w_{j_1t_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(2)}w_{t_2t_3}^k)\widehat{v}(h_{(3)}) \\
& = \sum_{k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{t_3j_1}^k)][h_{(3)}w_{j_1t_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(4)}w_{t_2t_3}^k)\widehat{v}(h_{(5)}).
\end{aligned}$$

On the other hand by Lemma 5.3 for any $h \in H$,

$$\begin{aligned}
& [\rho(a)\rho^\infty(x)(x^* \otimes 1^0)v](h) = [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_{\rho^\infty} x]x^*\widehat{v}(h_{(3)}) \\
& = N \sum_{i,j,k} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(3)}w_{ji}^k \cdot_{\rho^\infty} \widehat{w}(\tau)][e_{(1)} \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(e_{(2)})\widehat{v}(h_{(4)}) \\
& = N \sum_{i,j,k,i_1} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)}w_{ji_1}^k)\widehat{w}(\tau)(1 \rtimes_\rho S(h_{(4)}w_{i_1i}^k))(1 \rtimes_\rho e_{(1)}) \\
& \quad \times \widehat{w}(\tau)\widehat{V}^*(e_{(2)})\widehat{v}^*(e_{(3)})\widehat{v}(h_{(5)}) \\
& = N \sum_{i,j,k,i_1} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)}w_{ji_1}^k)\tau(S(h_{(4)}w_{i_1i}^k)e_{(1)})\widehat{w}(\tau)\widehat{V}^*(e_{(2)}) \\
& \quad \times \widehat{v}^*(e_{(3)})\widehat{v}(h_{(5)}) \\
& = N \sum_{i,j,k,i_1,i_2,i_3} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)}w_{ji_1}^k)\widehat{w}(\tau)\tau(S(h_{(6)}w_{i_3i}^k)e_{(1)}) \\
& \quad \times \widehat{V}^*(h_{(4)}w_{i_1i_2}^k S(h_{(5)}w_{i_2i_3}^k)e_{(2)})\widehat{v}^*(e_{(3)})\widehat{v}(h_{(7)}) \\
& = N \sum_{i,j,k,i_1,i_2} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)}w_{ji_1}^k)\widehat{w}(\tau)\tau(S(h_{(5)}w_{i_2i}^k)e_{(1)}) \\
& \quad \times \widehat{V}^*(h_{(4)}w_{i_1i_2}^k)\widehat{v}^*(e_{(2)})\widehat{v}(h_{(6)}) \\
& \\
& = N \sum_{i,j,k,i_1,i_2,t,t_1} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)}w_{ji_1}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(4)}w_{i_1i_2}^k) \\
& \quad \times \tau(S(h_{(7)}w_{t_1i}^k)e_{(1)})\widehat{v}^*(h_{(5)}w_{i_2t}^k S(h_{(6)}w_{tt_1}^k)e_{(2)})\widehat{v}(h_{(8)}) \\
& = N \sum_{i,j,k,i_1,i_2,t} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)}w_{ji_1}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(4)}w_{i_1i_2}^k) \\
& \quad \times \tau(S(h_{(6)}w_{ti}^k)e)\widehat{v}^*(h_{(5)}w_{i_2t}^k)\widehat{v}(h_{(7)}) \\
& = \sum_{j,k,i_1,i_2,t} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)}w_{ji_1}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(4)}w_{i_1i_2}^k)\widehat{v}^*(h_{(5)}w_{i_2t}^k) \\
& \quad \times \widehat{v}(h_{(6)}) \\
& = \sum_{j,k,i_1,i_2,t} d_k [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)][h_{(3)}w_{ji_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(4)}w_{i_2t}^k)\widehat{v}(h_{(5)}).
\end{aligned}$$

Therefore, we obtain the conclusion. \square

Lemma 10.6. *With the above notations and assumptions, for any $a \in F$,*

$$||xa - ax|| < L\epsilon + L \max_{i,j,k} ||\sigma([S(w_{ij}^k) \cdot_\rho a]) - \rho([S(w_{ij}^k) \cdot_\rho a])||,$$

where $L = \sum_{i,j,k} d_k ||\text{id} \otimes w_{ij}^k||$.

Proof. For any $a \in F$

$$\begin{aligned}
xax^* &= N \sum_{i,j,k} d_k \widehat{v}(w_{ij}^k) [w_{ji}^k \cdot_{\rho^\infty} \widehat{w}(\tau)] a [e_{(1)} \cdot_{\rho^\infty} \widehat{w}(\tau)] \widehat{v}^*(e_{(2)}) \\
&= N \sum_{i,j,k,i_1} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) \widehat{w}(\tau) (1 \rtimes_\rho S(w_{i_1 i}^k)) (a \rtimes_\rho e_{(1)}) \widehat{w}(\tau) \widehat{V}^*(e_{(2)}) \widehat{v}^*(e_{(3)}) \\
&= N \sum_{i,j,k,i_1,i_2} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) \widehat{w}(\tau) ([S(w_{i_2 i}^k) \cdot_\rho a] \rtimes_\rho S(w_{i_1 i_2}^k) e_{(1)}) \widehat{w}(\tau) \widehat{V}^*(e_{(2)}) \\
&\quad \times \widehat{v}^*(e_{(3)}) \\
&= N \sum_{i,j,k,i_1,i_2} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) [S(w_{i_2 i}^k) \cdot_\rho a] \tau(S(w_{i_1 i_2}^k) e_{(1)}) \widehat{w}(\tau) \widehat{V}^*(e_{(2)}) \widehat{v}^*(e_{(3)}) \\
&= N \sum_{i,j,k,i_1,i_2,t,t_1} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) [S(w_{i_2 i}^k) \cdot_\rho a] \widehat{w}(\tau) \widehat{V}^*(w_{i_1 t}^k S(w_{t t_1}^k) e_{(2)}) \\
&\quad \times \tau(S(w_{t_1 i_2}^k) e_{(1)}) \widehat{v}^*(e_{(3)}) \\
&= N \sum_{i,j,k,i_1,i_2,t} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) [S(w_{i_2 i}^k) \cdot_\rho a] \widehat{w}(\tau) \widehat{V}^*(w_{i_1 t}^k) \tau(S(w_{t i_2}^k) e_{(1)}) \widehat{v}^*(e_{(2)}) \\
&= N \sum_{i,j,k,i_1,i_2,t,s,s_1} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) [S(w_{i_2 i}^k) \cdot_\rho a] \widehat{w}(\tau) \widehat{V}^*(w_{i_1 t}^k) \tau(S(w_{s_1 i_2}^k) e_{(1)}) \\
&\quad \times \widehat{v}^*(w_{ts}^k S(w_{ss_1}^k) e_{(2)}) \\
&= N \sum_{i,j,k,i_1,i_2,t,s} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) [S(w_{i_2 i}^k) \cdot_\rho a] \widehat{w}(\tau) \widehat{V}^*(w_{i_1 t}^k) \tau(S(w_{s i_2}^k) e) \widehat{v}^*(w_{ts}^k) \\
&= \sum_{i,j,k,i_1,i_2,t} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) [S(w_{i_2 i}^k) \cdot_\rho a] \widehat{w}(\tau) \widehat{V}^*(w_{i_1 t}^k) \widehat{v}^*(w_{ti_2}^k) \\
&= \sum_{i,j,k,i_1,i_2,t} d_k \widehat{v}(w_{ij}^k) \widehat{V}(w_{ji_1}) \widehat{w}(\tau) [S(w_{i_2 i}^k) \cdot_\rho a] \widehat{V}^*(w_{i_1 t}^k) \widehat{v}^*(w_{ti_2}^k) \\
&= \sum_{i,k,i_2} d_k (\text{id} \otimes w_{ii_2}^k) (v \rho^\infty(\widehat{w}(\tau)) \rho([S(w_{i_2 i}^k) \cdot_\rho a]) v^*).
\end{aligned}$$

Hence

$$\begin{aligned}
&\|xax^* - \sum_{i,k,i_2} d_k (\text{id} \otimes w_{ii_2}^k) (v \rho^\infty(\widehat{w}(\tau)) v^* \sigma([S(w_{i_2 i}^k) \cdot_\rho a]))\| \\
&= \|\sum_{i,k,i_2} d_k (\text{id} \otimes w_{ii_2}^k) (v \rho^\infty(\widehat{w}(\tau)) [\rho([S(w_{i_2 i}^k) \cdot_\rho a]) v^* - v^* \sigma([S(w_{i_2 i}^k) \cdot_\rho a])])\| \\
&\leq \sum_{i,k,i_2} d_k \|\text{id} \otimes w_{ii_2}^k\| \|\rho([S(w_{i_2 i}^k) \cdot_\rho a]) v^* - v^* \sigma([S(w_{i_2 i}^k) \cdot_\rho a])\| \\
&\leq \sum_{i,k,i_2} d_k \|\text{id} \otimes w_{ii_2}^k\| \epsilon = L \epsilon
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{i,k,i_2} d_k (\text{id} \otimes w_{ii_2}^k) (v \rho^\infty(\widehat{w}(\tau)) v^* \rho([S(w_{i_2 i}^k) \cdot_\rho a])) \\
&= \sum_{i,k,i_2,t} d_k (v \rho^\infty(\widehat{w}(\tau)) v^*) \widehat{v}^*(w_{it}^k) [w_{ti_2}^k S(w_{i_2 i}^k) \cdot_\rho a] \\
&= \sum_{i,k} d_k (v \rho^\infty(\widehat{w}(\tau)) v^*) \widehat{v}^*(w_{ii}^k) a = N (\text{id} \otimes e) (v \rho^\infty(\widehat{w}(\tau)) v^*) a.
\end{aligned}$$

We recall that $\rho_1 = \text{Ad}(v) \circ \rho$, $u = (v \otimes 1^0)(\rho \otimes \text{id})(v)(\text{id} \otimes \Delta^0)(v^*)$ and that (ρ_1, u) is a twisted coaction of H^0 on A which is exterior equivalent to ρ . Then by Lemmas 5.5 and 5.6,

$$N(\text{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))v^*) = N[e \cdot_{\rho_1, u} \widehat{w}(\tau)] = 1.$$

Hence

$$\sum_{i, k, i_2} d_k(\text{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^* \rho([S(w_{i_2 i}^k) \cdot_\rho a])) = a.$$

It follows that

$$\begin{aligned} & \|xax^* - a\| \\ &= \|xax^* - \sum_{i, k, i_2} d_k(\text{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^* \sigma([S(w_{i_2 i}^k) \cdot_\rho a]))\| \\ &+ \sum_{i, k, i_2} d_k(\text{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^* \sigma([S(w_{i_2 i}^k) \cdot_\rho a])) \\ &- \sum_{i, k, i_2} d_k(\text{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^* \rho([S(w_{i_2 i}^k) \cdot_\rho a]))\| \\ &< L\epsilon + \sum_{i, k, i_2} d_k \|\text{id} \otimes w_{ii_2}^k\| \|\sigma([S(w_{i_2 i}^k) \cdot_\rho a]) - \rho([S(w_{i_2 i}^k) \cdot_\rho a])\| \\ &< L\epsilon + \sum_{i, j, k} d_k \|\text{id} \otimes w_{ij}^k\| \max_{i, j, k} \|\sigma([S(w_{ij}^k) \cdot_\rho a]) - \rho([S(w_{ij}^k) \cdot_\rho a])\| \\ &< L\epsilon + L \max_{i, j, k} \|\sigma([S(w_{ij}^k) \cdot_\rho a]) - \rho([S(w_{ij}^k) \cdot_\rho a])\|, \end{aligned}$$

where $L = \sum_{i, j, k} d_k \|\text{id} \otimes w_{ij}^k\|$. Then we obtain the conclusion. \square

By Lemmas 10.3, 10.4, 10.5 and 10.6, we obtain Lemma 10.2. We note that the constant positive number L in the above proofs, does not depend on coactions ρ and σ but depends on only H^0 . Also, we note that if a coaction ρ of H^0 on A has the Rohlin property, then a coaction $(\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}$ of H^0 on A has also the Rohlin property for any automorphism α of A .

Theorem 10.7. *Let A be a separable unital C^* -algebra and let ρ and σ be coactions of a finite dimensional C^* -Hopf algebra H^0 on A with the Rohlin property. We suppose that ρ and σ are approximately unitarily equivalent. Then there is an approximately inner automorphism θ such that*

$$\sigma = (\theta \otimes \text{id}) \circ \rho \circ \theta^{-1}.$$

Proof. We shall show this theorem in the same strategy as in the proof of [4, Theorem 3.5]. We choose an increasing family $\{F_n\}_{n=0}^\infty$ of finite subsets of A whose union is dense in A . By induction using Lemma 10.2, we can construct an increasing family $\{G_n\}_{n=0}^\infty$ of finite subsets of A whose union is dense in A , a sequence $\{x_n\}$ of unitary elements in A and a family of coactions ρ_{2n}, σ_{2n+1} , $n = 0, 1, 2, \dots$, of

H^0 on A satisfying the following conditions:

$$\begin{aligned}
\rho_0 &= \rho, \quad \sigma_1 = \sigma \\
\rho_{2n+2} &= \text{Ad}(x_{2n} \otimes 1^0) \circ \rho_{2n} \circ \text{Ad}(x_{2n}^*), \quad n = 0, 1, 2, \dots, \\
\sigma_{2n+1} &= \text{Ad}(x_{2n-1} \otimes 1^0) \circ \sigma_{2n-1} \circ \text{Ad}(x_{2n-1}^*), \quad n = 1, 2, \dots, \\
F_{2n}^1 &= \bigcup_{i,j,k} [S(w_{ij}^k) \cdot_{\sigma_{2n+1}} F_{2n}], \quad n = 0, 1, \dots, \\
F_{2n+1}^1 &= \bigcup_{i,j,k} [S(w_{ij}^k) \cdot_{\sigma_{2n+2}} F_{2n+1}], \quad n = 0, 1, \dots, \\
G_0 &= F_0 \cup F_0^1 \\
G_{2n+1} &= G_{2n} \cup F_{2n+1} \cup F_{2n+1}^1, \quad n = 0, 1, \dots, \\
G_{2n+2} &= G_{2n+1} \cup F_{2n+2} \cup F_{2n+2}^1, \quad n = 0, 1, \dots, \\
\|\sigma_{2n+1}(a) - \rho_{2n+2}(a)\| &< \frac{1}{2^{2n}}, \quad a \in G_{2n}, \quad n = 0, 1, \dots, \\
\|\sigma_{2n+3}(a) - \rho_{2n+2}(a)\| &< \frac{1}{2^{2n+1}}, \quad a \in G_{2n+1}, \quad n = 0, 1, \dots, \\
\|x_{2n+1}a - ax_{2n+1}\| &< \frac{1}{2^{2n+1}} + L \max_{i,j,k} \|\rho_{2n+2}([S(w_{ij}^k) \cdot_{\sigma_{2n+1}} a]) \\
&\quad - \sigma_{2n+1}([S(w_{ij}^k) \cdot_{\sigma_{2n+1}} a])\| < \frac{2+L}{2^{2n}}, \quad a \in G_{2n}, \quad n = 0, 1, \dots, \\
\|x_{2n}a - ax_{2n}\| &< \frac{1}{2^{2n}} + L \max_{i,j,k} \|\sigma_{2n+1}([S(w_{ij}^k) \cdot_{\rho_{2n}} a]) \\
&\quad - \rho_{2n}([S(w_{ij}^k) \cdot_{\rho_{2n}} a])\| < \frac{2+L}{2^{2n-1}}, \quad a \in G_{2n-1}, \quad n = 1, 2, \dots,
\end{aligned}$$

In the same way as in the proof of [4, Theorem 3.5], we can obtain the conclusion. \square

In the rest of this section, we shall study coactions having the Rohlin property of a finite dimensional C^* -Hopf algebra on a UHF-algebra of type N^∞ . Let A be a UHF-algebra of type N^∞ . Let $M_n(\mathbf{C})$ be the $n \times n$ -matrix algebra over \mathbf{C} and $\{f_{ij}\}$ a system of matrix units of $M_n(\mathbf{C})$.

Lemma 10.8. *Let ρ be a unital homomorphism of A to $A \otimes M_n(\mathbf{C})$ and ρ_* the homomorphism of $K_0(A)$ to $K_0(A \otimes M_n(\mathbf{C}))$ induced by ρ . Then $\rho_*([\frac{1}{N^l}]) = n[\frac{1}{N^l}]$ for any $l \in \mathbf{N} \cup \{0\}$.*

Proof. Since $\rho(1) = 1 \otimes I_n$, $\rho_*([1]) = [1 \otimes I_n] = n[1 \otimes f_{11}] = n[1]$. Hence $N\rho_*([\frac{1}{N}]) = \rho_*([1]) = n[1]$. Since $K_0(A) = \mathbf{Z}[\frac{1}{N}]$ is torsion-free, $\rho_*([\frac{1}{N}]) = n[\frac{1}{N}]$. \square

Lemma 10.9. *Let ρ be a unital homomorphism of A to $A \otimes M_n(\mathbf{C})$. Then there is a sequence $\{u_k\}$ of unitary elements in $A \otimes M_n(\mathbf{C})$ such that for any $x \in A$*

$$\rho(x) = \lim_{k \rightarrow \infty} u_k(x \otimes I_n)u_k^*.$$

Proof. Modifying the proof of Blackadar [1, 7.7 Exercises and Problems] we can prove this lemma. Let $\{A_k\}$ be a increasing sequence of full matrix algebras over \mathbf{C} with $\overline{\bigcup_k A_k} = A$. Let $\{e_{ij}\}$ be a system of matrix units of A_k . Since A has the cancellation property, by Lemma 10.8, $\rho(e_{11}) \sim e_{11} \otimes I_n$ in $A \otimes M_n(\mathbf{C})$. Hence there is a partial isometry $w \in A \otimes M_n(\mathbf{C})$ such that

$$w^*w = E_{11}, \quad ww^* = \rho(e_{11}),$$

where $E_{ij} = e_{ij} \otimes I_n$ for any i, j . Let $u_k = \sum_i \rho(e_{i1})wE_{1i}$. Then u_k is a unitary element in $A \otimes M_n(\mathbf{C})$ by easy computations. Let $x \in A_k$. Then we can write

that $x = \sum_{i,j} \lambda_{ij} e_{ij}$, where $\lambda_{ij} \in \mathbf{C}$. Hence by easy computations, we can see that $\rho(x) = u_k(x \otimes I_n)u_k^*$. Since $\overline{\bigcup_k A_k} = A$, we obtain that for any $x \in A$, $\rho(x) = \lim_{k \rightarrow \infty} u_k(x \otimes I_n)u_k^*$ by routine computations. \square

Lemma 10.10. *Let ρ be a unital homomorphism of A to $A \otimes H^0$, where H^0 is a finite dimensional C^* -algebras. Then there is a sequence $\{u_k\}$ of unitary elements in $A \otimes H^0$ such that for any $x \in A$*

$$\rho(x) = \lim_{k \rightarrow \infty} u_k(x \otimes 1)u_k^*.$$

Proof. Let $\{p_l\}$ be a family of minimal central projections in H^0 . For any l and $x \in A$, let

$$\rho_l(x) = \rho(x)(1 \otimes p_l).$$

Then by Lemma 10.9, there is a sequence $\{u_k^{(l)}\}$ of unitary elements in $A \otimes p_l H^0$ such that $\rho_l(x) = \lim_{k \rightarrow \infty} u_k^{(l)}(x \otimes p_l)u_k^{(l)*}$ for any $x \in A$. Let $u_k = \oplus_l u_k^{(l)}$. Then we can see that $\{u_k\}$ is a desired sequence by easy computations. \square

Corollary 10.11. *Let H be a finite dimensional C^* -Hopf algebra with dimension N and let A be a UHF-algebra of type N^∞ . Let ρ be a coaction of H^0 on A with the Rohlin property constructed in Section 7. Then for any coaction σ of H^0 on A with the Rohlin property, there is an approximately inner automorphism θ of A such that*

$$\sigma = (\theta \otimes \text{id}) \circ \rho \circ \theta^{-1}.$$

Proof. By Lemma 10.10, σ is approximately unitarily equivalent to ρ . Hence by Theorem 10.7, we obtain the conclusion. \square

11. APPENDIX

In the previous paper [8], we introduced the Rohlin property for weak coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra. In this section, we shall show that if there is a weak coaction with the Rohlin property in the sense of [8] of a finite dimensional C^* -Hopf algebra H on a unital C^* -algebra A , then H is commutative. Recall that a weak coaction ρ of H on A has the Rohlin property in the sense of [8] if there is a monomorphism π of H into A_∞ such that for any $h \in H$, $\rho^\infty(\pi(h)) = \pi(h_{(1)}) \otimes h_{(2)}$. Let $\{w_{ij}^k\}$ be a system of comatrix units of H .

Lemma 11.1. *With the above notations, $(H \otimes 1)\Delta(H) = H \otimes H$.*

Proof. For any i, j, k , $\Delta(w_{ij}^k) = \sum_t w_{it}^k \otimes w_{tj}^k$. Since $\sum_i w_{it}^{k*} w_{is}^k = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$ for any k by [10, Theorem 2.2, 2], we can obtain that

$$\sum_i (w_{it}^{k*} \otimes 1) \Delta(w_{ij}^k) = \sum_{i,s} w_{it}^{k*} w_{is}^k \otimes w_{sj}^k = 1 \otimes w_{tj}^k.$$

Thus we obtain the conclusion. \square

Lemma 11.2. *With the above notations, let ρ be a weak coaction of H on A with the Rohlin property in the sense of [8]. Then $(A \otimes 1)\rho(A) = A \otimes H$.*

Proof. Since ρ has the Rohlin property in the sense of [8], there is a monomorphism π of H into A_∞ . First, we show that $(A^\infty \otimes 1)\rho^\infty(A^\infty) = A^\infty \otimes H$. Since $\rho^\infty \circ \pi = (\pi \otimes \text{id}) \circ \Delta$,

$$\begin{aligned} (\pi(H) \otimes 1)\rho^\infty(\pi(H)) &= (\pi(H) \otimes 1)(\pi \otimes \text{id})(\Delta(H)) \\ &= (\pi \otimes \text{id})((H \otimes 1)\Delta(H)) = \pi(H) \otimes H \end{aligned}$$

by Lemma 11.1. Since $1 \otimes w_{ij}^k \in \pi(H) \otimes H$, $1 \otimes w_{ij}^k \in (A^\infty \otimes 1)\rho^\infty(A^\infty)$. Thus we can see that $(A^\infty \otimes 1)\rho^\infty(A^\infty) = A^\infty \otimes H$. For any $x \in A \otimes H$, there are $a_1, \dots, a_n, b_1, \dots, b_n \in A^\infty$ such that $x = \sum_{i=1}^n (a_i \otimes 1)\rho^\infty(b_i)$. That is,

$$\|x - \sum_{i=1}^n (a_i^{(k)} \otimes 1)\rho^\infty(b_i^{(k)})\| \rightarrow 0 \quad (k \rightarrow \infty),$$

where $a_i = (a_i^{(k)}), b_i = (b_i^{(k)})$ and $a_i^{(k)}, b_i^{(k)} \in A$ for any k, i . Hence $x \in \overline{(A \otimes 1)\rho(A)}$. \square

Proposition 11.3. *Let ρ be a weak coaction of H on A with the Rohlin property in the sense of [8] and π a monomorphism of H to A_∞ . Then $\rho^\infty(\pi(H)) \subset (A \otimes H)' \cap (A^\infty \otimes H)$.*

Proof. Let $a, b \in A$ and $h \in H$. Then

$$\begin{aligned} \rho^\infty(\pi(h))(a \otimes 1)\rho(b) &= (\pi(h_{(1)}) \otimes h_{(2)})(a \otimes 1)\rho(b) = (a\pi(h_{(1)}) \otimes h_{(2)})\rho(b) \\ &= (a \otimes 1)\rho^\infty(\pi(h))\rho(b) = (a \otimes 1)\rho^\infty(\pi(h)b) \\ &= (a \otimes 1)\rho(b)\rho^\infty(\pi(h)). \end{aligned}$$

Therefore we obtain the conclusion by Lemma 11.2. \square

Proposition 11.4. *Let ρ be a weak coaction of H on A with the Rohlin property. Let x be any element in $A \otimes H$. Then for any $h \in H$, $(1 \otimes h)x = x(1 \otimes h)$.*

Proof. Let π be a monomorphism of H into A_∞ such that $\rho^\infty(\pi(h)) = \pi(h_{(1)}) \otimes h_{(2)}$ for any $h \in H$. By the proof of Lemma 11.2, we can see that

$$1 \otimes H \subset \pi(H) \otimes H = (\pi(H) \otimes 1)\rho^\infty(\pi(H)).$$

Hence it suffices to show that for any $h \in H$,

- (1) $(\pi(h) \otimes 1)x = x(\pi(h) \otimes 1)$,
- (2) $\rho^\infty(\pi(h))x = x\rho^\infty(\pi(h))$.

Indeed, since $x \in A \otimes H$ and $\pi(h)$ commute with any element in A for any $h \in H$, we obtain (1). Also, we can obtain (2) by Proposition 11.3 \square

Corollary 11.5. *Let A be a unital C^* -algebra and H a finite dimensional C^* -Hopf algebra. If there is a weak coaction of H on A with the Rohlin property in the sense of [8], then H is commutative.*

Proof. This is immediate by Proposition 11.4. \square

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